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A New Survival Assumption for GEI Economies*

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Abstract This paper introduces the concept of "robust survival" as a sufficient condition for the existence of equilibrium in a general equilibrium model with incomplete markets (GEI). The robust survival condition is weaker than the GEI irreducibility condition proposed by Gottardi and Hens (1996) in the GEI model, and it is less affected by the degree of market incompleteness. Unlike the GEI irreducibility condition of Gottardi and Hens (1996), the robust survival condition provides an explanation for the existence of a GEI equilibrium in which an agent can consume with the minimum expenditure on feasible consumptions. This condition can be used to evaluate the impact of financial innovation on the welfare of the poor. When an economy passes the robust survival condition but fails the GEI irreducibility condition, some agents may be "poor" in pre-innovation equilibrium. In this case, we can apply the GEI irreducibility condition to the post-innovation economy to determine whether financial innovation makes the invisible hand benevolent towards the poor.

Keywords Incomplete markets, GEI irreducibility, robust survival, financial innovation, state-contingent poverty

JEL Classification D52, C62, G10

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1. INTRODUCTION

The irreducibility assumption of (McKenzie, 1959, 1981) represents a survival condition for complete-market general equilibrium models. It ensures that every agent can afford consumptions in competitive equilibrium that are less valuable than the initial endowment. The cheaper consumption property is one of the main characteristics of competitive equilibrium in classical general equilibrium theory. Gottardi and Hens (1996) extend the irreducibility condition to the general equilibrium model with incomplete markets (GEI model). If an economy fulfills their GEI irreducibility condition, every agent possesses cheaper consumptions in equilibrium of the GEI economy, implying that all agents live above the subsistence income.

The cheaper consumption property is not as prevalent in an incomplete-market economy as it is in a complete-market economy, due to the limited capability of asset markets to transfer income across time and contingencies.[1] Incomplete markets may not offer a full set of financial assets that agents need to hold to finance cheaper consumptions in each state of the future. To illustrate this, let's consider a two-agent, two-period, three-state GEI model with one good and two assets. The first asset pays one unit of the good except in the first state, and the second asset pays one unit of the good except in the second state. Suppose the first agent is endowed with a positive amount of the good in each state, while the second agent is endowed with nothing except in the third state. In this case, the second agent, being unable to trade, chooses not to engage in any transactions, which leads to a no-trade equilibrium. As demonstrated in Example 3.1 later, this economy is not GEI irreducible because the second agent, who is considered "poor" in the first and second states, cannot offer any income transfer that improves the welfare of the first agent. [2] This example highlights that GEI irreducibility can fail when risk-sharing opportunities are severely restricted due to market incompleteness. If financial innovation were to make asset markets complete in the economy, the post-innovation economy would become GEI irreducible. Hence, the degree of market incompleteness can critically impact the validity of the GEI irreducibility condition.

The paper introduces the concept of "robust survival" as a sufficient condition for the existence of equilibrium in GEI economies where some agents can consume at the subsistence level. The GEI model is said to satisfy robust sur-

[1] This property always holds in complete markets where every agent has strictly monotonic preferences and possesses positive initial endowments.

[2] Agents are said to be poor in a state if they consume at a subsistence level in that state. The formal definition of poverty is introduced in Section 2 of the paper.

vival if, for each feasible allocation where each consumption are budget-feasible, there exist agents with non-optimal consumptions who are allowed to choose a better alternative that can be perturbed within slightly perturbed budget sets. Robust survival is weaker than the GEI irreducibility of Gottardi and Hens (1996) and less affected by the degree of market incompleteness. Unlike the GEI irreducibility condition of Gottardi and Hens (1996), the robust survival condition provides an explanation for the existence of GEI equilibrium in which an agent can consume with the minimum expenditure on feasible consumptions. The new survival condition is the first attempt to investigate the GEI model in which agents may not be able to afford cheaper consumptions in equilibrium. Robust survival includes the GEI irreducibility condition as a special case, even when the cheaper consumption property holds. This point is illustrated in Example 3.2, where the cheaper consumption property holds in an economy that is not GEI irreducible. The robust survival condition can be used to evaluate the impact of financial innovation on the welfare of the poor. When an economy passes the robust survival condition but fails the GEI irreducibility condition, some agents may be ‘poor’ in certain states in a pre-innovation equilibrium. In this case, we can apply the GEI irreducibility condition to the post-innovation economy to determine whether financial innovation makes the invisible hand benevolent towards the poor.

The introduction of new securities expands risk-sharing opportunities among individuals.[3] However, financial innovation may not always be beneficial to the economy. For example, Hart (1975) and Elul (1995) demonstrate that it can make every agent worse off in cases where the cheaper consumption property holds.[4] Moreover, financial innovation can lead to even more adverse consequences in a GEI economy that lacks the cheaper consumption property. As illustrated later, when financial innovation prevents the state-contingent poor from accessing cheaper consumptions, it undermines the functioning of the “invisible hand,” potentially leading to a failure to achieve equilibrium in the post-innovation stage. Therefore, we need to examine how financial innovation impacts market outcomes in the presence of agents that experience state-contingent poverty.

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[3]Financial innovation can give rise to redundant assets, whose payoffs can be replicated by a portfolio of other assets. For example, the put-call parity implies that either the underlying asset, call option, put option, or zero-coupon bonds are redundant. Hahn and Won (2012) extend the concept of GEI irreducibility to constrained asset markets, where agents face restrictions on their portfolio holdings.

[4]They assume that agents have strictly positive endowments. In this case, the cheaper consumption property trivially holds in the pre- and post-innovation economies.

Conventional comparative statics based on differential calculus cannot be employed to analyze the effects of financial innovation because it alters the asset structure and, thus, the dimension of marketed spaces. Notably, comparative statics analysis can be conducted to examine the welfare change of the state-contingent poor between pre- and post-innovation equilibria by sequentially applying the concepts of robust survival and GEI irreducibility. To illustrate this, let us consider a GEI economy that satisfies robust survival but is not GEI irreducible. In this scenario, let us assume that the economy lacks the cheaper consumption property and is prepared to introduce a new asset to enhance risk-sharing opportunities among the agents. The policy question at hand is whether financial markets function effectively in the post-innovation stage and, if so, whether the expanded risk-sharing opportunities benefit the state-contingent poor by enabling every agent to avoid state-contingent poverty. To answer this question, we can assess the impact of financial innovation on poverty by applying the GEI irreducibility test to the post-innovation economy. If financial innovation ensures the GEI irreducibility of the economy, it will eliminate state-contingent poverty. Therefore, the combination of robust survival and the GEI irreducibility condition provides a valuable framework for examining both the positive and normative aspects of financial innovation.

Financial innovation enhances risk-sharing opportunities in incomplete markets, which can alter agents' access to cheaper consumption options. Consequently, it can have implications for the existence of equilibrium in the post-innovation economy. One might conjecture that equilibrium is more likely to occur in the post-innovation economy. However, as illustrated in Example 5.2, financial innovation is not always advantageous for the existence of equilibrium when state-contingent poverty is present. Therefore, it is essential to address the issue of equilibrium existence before proceeding with comparative statics analysis on the welfare effect of financial innovation. To highlight the counterintuitive aspect of financial innovation, let us take Example 2 of Gottardi and Hens (1996) that deals with a two-asset, three-state economy. We create a single-asset economy by deleting the second asset in this example. The single-asset economy satisfies robust survival and has a no-trade equilibrium. However, it is not GEI irreducible due to the fact that agent 2 does not consume anything in the second state. Now, let us suppose that the second asset is introduced into the economy through financial innovation. The post-innovation economy corresponds to the one described in Example 2 of Gottardi and Hens (1996), which does not possess an equilibrium. When certain agents experience state-contingent poverty, financial innovation has the potential to cause the breakdown of the market system.

2. THE MODEL

We consider a two-period economy with I agents whose index set is denoted by $\mathcal{I} = \{1, 2, \ldots, I\}$. Financial markets are open in the first period (period 0) while markets for consumption goods are open in the second period (period 1). There are finite states of the world in the second period, whose index set is denoted by $\mathcal{S} = \{1, \ldots, S\}$. Agents consume L consumption goods in each state $s \in \mathcal{S}$. Since consumption is available only in the second period, the total number of commodities equals $\ell := LS$, implying that $\mathbb{R}^\ell$ becomes the commodity space of the economy.

Agent $i \in \mathcal{I}$ chooses consumption bundles in his consumption set $X_i := \mathbb{R}^\ell_+$, is initially endowed with $e_i \in \mathbb{R}^\ell_+$, and has a preference relation $\succ_i$ on $\mathbb{R}^\ell_+$. The preference relation $\succ_i$ induces a correspondence P_i on $\mathbb{R}^\ell_+$ defined by $P_i(x_i) = \{x'_i \in \mathbb{R}^\ell_+ : x'_i \succ_i x_i\}$ for each $x_i \in \mathbb{R}^\ell_+$. Agents can transfer income intertemporally by holding J financial assets whose index set is denoted by $\mathcal{J} = \{1, 2, \ldots, J\}$. Each asset $j \in \mathcal{J}$ pays the $r_j(s)$ monetary units in state s. The payoff of J assets in state s is given by the J-dimensional row vector $r(s) = (r_j(s))_{j \in \mathcal{J}}$, whereas the payoff of asset j is summarized as the S-dimensional column vector $r_j = (r_j(s))_{s \in \mathcal{S}}$. The asset (payoff) structure is described by the $S \times J$ matrix $R = [(r(s))_{s \in \mathcal{S}}]$. We assume that $J \leq S$. Each agent is allowed to choose a portfolio in his portfolio set $\Theta_i := \mathbb{R}^J$.

The sets $X := \prod_{i \in \mathcal{I}} X_i = \mathbb{R}^{\ell I}_+$ and $\Theta := \prod_{i \in \mathcal{I}} \Theta_i = \mathbb{R}^{JI}$ represent the set of feasible consumption and portfolio allocations, respectively. Let A denote the set of attainable allocations, i.e.,

$$A = \{(x, \theta) \in X \times \Theta : \textstyle\sum_{i \in \mathcal{I}} (x_i - e_i) = 0, \ \sum_{i \in \mathcal{I}} \theta_i = 0\}.$$

and let $A_X = \{x \in X : \sum_{i \in \mathcal{I}} (x_i - e_i) = 0\}$, which is the set of market-clearing consumption allocations.

For a pair (p, q) in $\mathbb{R}^\ell \times \mathbb{R}^J$ and a point y in $\mathbb{R}^\ell$, we follow the notational convention:

$$p \,\square\, y := \begin{bmatrix} 0 \\ p \,\square_1\, y \end{bmatrix}, \quad W(q) := \begin{bmatrix} -q \\ R \end{bmatrix},$$

where $p \,\square_1\, y$ indicates the S-dimensional column vector $(p(s) \cdot y(s))_{s \in \mathcal{S}}$. The zero in $p \,\square\, y$ is intended to indicate the fact that no consumption arises in the first period. The open budget set of agent i at prices (p, q) is defined by

$$\mathcal{B}_i(p, q) = \left\{ (x_i, \theta_i) \in \mathbb{R}^\ell_+ \times \mathbb{R}^J : p \,\square\, (x_i - e_i) \ll W(q) \cdot \theta_i \right\}.^5$$

[5]For two vectors v and v' in an Euclidean space, $v \geq v'$ indicates that $v - v' \in \mathbb{R}^\ell_+$; $v > v'$ indicates that $v \geq v'$ and $v \neq v'$; $v \gg v'$ indicates that $v - v' \in \mathbb{R}^\ell_{++}$

and his budget set is defined by $c\ell\mathcal{B}_i(p,q) = c\ell[\mathcal{B}_i(p,q)]$.[6] An element (x_i, θ_i) is $\succ_i$-maximal in $c\ell\mathcal{B}_i(p,q)$ if $(P_i(x_i)\times\mathbb{R}^J)\cap c\ell\mathcal{B}_i(p,q) = \emptyset$. Let $\mathcal{E} = \langle(\mathbb{R}^\ell_+, \succ_i, e_i, \mathbb{R}^J)_{i\in\mathcal{I}}, R\rangle$ denote the economy described above.

Definition 2.1: A *competitive equilibrium* of economy $\mathcal{E}$ is a vector $(p^*, q^*, x^*, \theta^*) \in \mathbb{R}^\ell \times \mathbb{R}^J \times \mathbb{R}^{\ell I}_+ \times \mathbb{R}^{JI}$ such that

(*i*) $(x_i^*, \theta_i^*) \in c\ell\mathcal{B}_i(p^*, q^*), \forall i \in \mathcal{I}$,

(*ii*) $\left(P_i(x_i^*)\times\mathbb{R}^J\right)\cap c\ell\mathcal{B}_i(p^*,q^*) = \emptyset, \forall i \in \mathcal{I}$,

(*iii*) $(x^*, \theta^*) \in A$.

We make the following assumptions for each $i \in \mathcal{I}$.

(A1) $\succ_i$ is irreflexive on $\mathbb{R}^\ell_+$.

(A2) $\succ_i$ is continuous, monotonic, and convex on $\mathbb{R}^\ell_+$.[7]

(A3) R has rank J and there exists $\theta \in \mathbb{R}^J$ with $R\cdot\theta > 0$.

(A4) $e_i \in \mathbb{R}^\ell_+ \setminus \{0\}$ and $\sum_{i\in\mathcal{I}} e_i \gg 0$.

Preferences need not be either transitive or complete. Assumption (A3) states that there is no redundant asset, and the asset structure can generate non-negative income in all states and positive income in certain states. The latter condition ensures that asset demand reaches no satiation, and holds true in the presence of risk-free assets. The first part of Assumption (A4) requires that each agent is self-subsistent. When each agent has monotone preferences, the self-subsistence condition is sufficient for the existence of equilibrium in complete-market economies. As illustrated in Gottardi and Hens (1996), market incompleteness may lead to the existence failure in the case that the initial endowments are not strictly positive. Survival conditions are discussed in the next section to make up for the insufficiency of Assumption (A4).

Here every agent is assumed to have the consumption set $\mathbb{R}^\ell_+$. The consumption set $\mathbb{R}^\ell_+$ can result from normalizing the subsistence consumption of each

[6]For a nonempty subset A of an Euclidean space, we denote the closure of A by $c\ell A$, and the interior of A by $intA$,.

[7]Let x_i be any point in $\mathbb{R}^\ell_+$. The preference ordering $\succ_i$ is monotonic if $x_i + v_i \in P_i(x_i)$ for every $v \in \mathbb{R}^\ell_+ \setminus \{0\}$, is continuous if both $P_i(x_i)$ and $\{x_i' \in \mathbb{R}^\ell_+ : x_i \succ_i x_i'\}$ are open, and is convex if $P_i(x_i)$ is convex.

agent in each state to zero. To see this, let X_i' denote the set of feasible consumptions that contains its unique lower bound $\underline{x}_i$ where $\underline{x}_i(s)$, the lower bound in state s, indicates the minimal consumption for subsistence or the threshold to poverty in state s for agent i. Since elements in X_i' are at least as great as $\underline{x}_i$, X_i' can be replaced by the normalized consumption set $X_i =: X_i' - \{\underline{x}_i\} = \mathbb{R}^{\ell}_{+}$. In this case, preferences are defined in X_i by translating the preferred sets by $\underline{x}_i$. Then the post-translation economy shares the same equilibrium prices with the pre-translation economy.

An asset price is arbitrage-free if it gives positive value to portfolios in $\mathbb{R}^J$ that generate positive income transfers to the second period. The set of arbitrage-free asset prices is denoted by

$$Q = \{q \in \mathbb{R}^J : q \cdot v > 0 \text{ for all } v \in \mathbb{R}^J \text{ with } R \cdot v > 0\}.$$

Following Gottardi and Hens (1996), we need the following assumption.

(A5) For each $(p,q) \in \mathbb{R}^{\ell}_{++} \times c\ell Q$, there exists an agent $i \in \mathcal{I}$ with $\zeta_i \in \mathbb{R}^J$ which satisfies $q \cdot \zeta_i < 0$ and $p \square_1 e_i + R \cdot \zeta_i \geq 0$.

Especially, Assumption (A5) ensures that there exists an agent who is able to build a portfolio to generate income necessary to exploit an arbitrage opportunity available at prices $q \in c\ell Q \setminus Q$. As q comes closer to $c\ell Q \setminus Q$, there exists an agent i whose optimal consumptions explode.

We provide the formal definition of cheaper consumptions.

Definition 2.2: Agent $i \in \mathcal{I}$ affords a *cheaper consumption* at price $(p,q) \in \mathbb{R}^{\ell}_{++} \times Q$ if $\mathcal{B}_i(p,q) \neq \emptyset$. Agent i is in *state-contingent poverty* (SC poverty, for short) at (p,q) if $\mathcal{B}_i(p,q) = \emptyset$. The economy has the *cheaper consumption property* at (p,q) if every agent affords a cheaper consumption at (p,q).

Agent i affords cheaper consumptions if the state-contingent income from the initial endowments and asset holdings is higher than the minimum expenditure on feasible consumptions at (p,q) in each state. He is in SC poverty if his income hits the minimum expenditure in some states. When there exists $v^\circ \in \mathbb{R}^J$ with $R \cdot v^\circ \gg 0$, Assumption (A5) implies that at least one agent $i \in \mathcal{I}$ affords cheaper consumptions at price $(p,q) \in \mathbb{R}^{\ell}_{++} \times c\ell Q$.

3. ROBUST SURVIVAL vs. GEI IRREDUCIBILITY

In this section, we provide a formal definition of robust survival. To do this, we let $\langle R \rangle$ denote the marketed subspace, i.e., $\langle R \rangle := \{w \in \mathbb{R}^S : w = R \cdot v$ for some $v \in \mathbb{R}^J\}$ and, for each $(p,q,x) \in \mathbb{R}^\ell_{++} \times Q \times A_X$, define a set of agents:

$$\left\{i \in \mathcal{I} : (x_i, \theta_i) \in c\ell\mathcal{B}_i(p,q) \text{ for some } \theta_i \in \mathbb{R}^J \text{ and } \left(P_i(x_i) \times \mathbb{R}^J\right) \cap c\ell\mathcal{B}_i(p,q) = \emptyset\right\}$$

The set $\mathcal{I}(p,q,x)$ indicates the set of agents that possess an optimal consumption in the allocation x. Robust survival is defined as following.

(A6) **(Robust Survival)** Let (p,q,x) be a point in $\mathbb{R}^\ell_{++} \times Q \times A_X$ with the following properties:

(6a) for some $i \in \mathcal{I}$, x_i is an optimal consumption at (p,q), i.e., $\mathcal{I}(p,q,x) \neq \emptyset$,

(6b) for some $i \in \mathcal{I}$, x_i is not an optimal consumption at (p,q), i.e., $\mathcal{I} \setminus \mathcal{I}(p,q,x) \neq \emptyset$, and

(6c) for each $i \in \mathcal{I}$, $p \,\square_1\, (x_i - e_i) \in \langle R \rangle$.

Let $(p^n, q^n, \varepsilon_n) \to (p,q,0)$ be a sequence in $\mathbb{R}^\ell_{++} \times Q \times \mathbb{R}^\ell_{++}$. Then for some $h \in \mathcal{I} \setminus \mathcal{I}(p,q,x)$, there exists (y_h, η_h) in $c\ell\mathcal{B}_h(p,q)$ with $y_h \succ_h x_h$ that admits a sequence $\{(y_h^n, \eta_h^n)\}$ in $(\mathbb{R}^\ell_+ - \{\varepsilon_n\}) \times \mathbb{R}^J$ such that $(y_h^n, \eta_h^n) \to (y_h, \eta_h)$ and

$$p^n \,\square\, (y_h^n - e_h) \leq W(q^n) \cdot \eta_h^n. \tag{1}$$

Let (p,q,x) be a non-equilibrium price-allocation pair such that (p,x) satisfies (6a), and one of the consumptions in x, say x_h, is not optimal. The robust survival condition is fulfilled if x_h is less preferred to a consumption choice y_h that admits perturbations satisfying the budget inequalities (1). The GEI model satisfies robust survival if, for each feasible allocation where each consumption are budget-feasible, there exist agents with non-optimal consumptions who are allowed to choose a better alternative that can be perturbed within slightly perturbed budget sets.

In particular, Assumption (A6) is fulfilled if, for each list (p,q,x) in $\mathbb{R}^\ell_{++} \times Q \times A_X$ that satisfies $\mathcal{I} \setminus \mathcal{I}(p,q,x) \neq \emptyset$ and $p \,\square_1\, (x_i - e_i) \in \langle R \rangle$ for each $i \in \mathcal{I}$, there exists $h \in \mathcal{I} \setminus \mathcal{I}(p,q,x)$ with $\mathcal{B}_h(p,q) \neq \emptyset$. This idea is built into the following slightly stronger version of robust survival.

(A6′) **(Strong Robust Survival)** Let (p,q,x,θ) be a point in $\mathbb{R}^\ell_{++} \times Q \times A$ that is not an equilibrium and satisfies $p \,\square_1\, (x_i - e_i) = R \cdot \theta_i$ for each $i \in \mathcal{I}$, and $(p^n, q^n, \varepsilon_n) \to (p,q,0)$ be a sequence in $\mathbb{R}^\ell_{++} \times Q \times \mathbb{R}^\ell_{++}$. Then for some $h \in \mathcal{I}$, there exists (y_h, η_h) in $c\ell\mathcal{B}_h(p,q)$ with $y_h \succ_h x_h$ that admits a sequence $\{(y_h^n, \eta_h^n)\}$ in $(\mathbb{R}^\ell_+ - \{\varepsilon_n\}) \times \mathbb{R}^J$ such that $(y_h^n, \eta_h^n) \to (y_h, \eta_h)$ and $p^n \,\square\, (y_h^n - e_h) \leq W(q^n) \cdot \eta_h^n$.

Condition (A6′) is a special case of (A6). The distinction between Assumptions (A6′) and (A6) is that (A6′) does not require the condition $\mathcal{I}(p,q,x) \neq \emptyset$. As demonstrated in the next section, the strong version of robust survival is weaker than the irreducibility condition of Gottardi and Hens (1996), and has an advantage in dealing with economies that lack the cheaper consumption property. For comparison, the GEI irreducibility of Gottardi and Hens (1996) is restated below.

(A7) (GEI Irreducibility) For any nontrivial partition $\{\mathcal{I}_1, \mathcal{I}_2\}$ of $\mathcal{I}$, any price $p \in \mathbb{R}^{\ell}_{++}$, and any $x \in A_X$ that satisfies $p \,\square_1\, (x_i - e_i) \in \langle R \rangle$ for each $i \in \mathcal{I}$, there exist $(z_1, \ldots, z_I) \in \mathbb{R}^{\ell I}$ and $(\phi_1, \ldots, \phi_I) \in \mathbb{R}^{JI}$ such that

(i) $e_i + z_i \in \mathbb{R}^{\ell}_{+}, \forall i \in \mathcal{I}_1$,

(ii) $x_i + z_i \succ_i x_i$ for each $i \in \mathcal{I}_2$,

(iii) $p \,\square_1\, z_i = R \cdot \phi_i$ for each $i \in \mathcal{I}$, and

(iv) $\sum_{i \in \mathcal{I}} z_i = 0$.

It is noted that, since R has full rank, (iii) and (iv) of GEI irreducibility imply that $\sum_{i \in \mathcal{I}} \phi_i = 0$.

We show that Assumption (A6′) is weaker than GEI irreducibility in incomplete markets, and discuss the relationship between GEI irreducibility and the cheaper consumption property. Several examples are provided below to discuss the advantage of robust survival over the GEI irreducibility condition.

Proposition 3.1: Under Assumptions (A1)–(A5), the GEI irreducibility implies Assumption (A6′), and therefore (A6).

PROOF : See Appendix. ■

Proposition 3.2: Suppose that Assumptions (A1)–(A5) hold and there exists $v^{\circ} \in \mathbb{R}^J$ with $R \cdot v^{\circ} \gg 0$. If economy $\mathcal{E}$ is GEI irreducible, then the cheaper consumption property holds in equilibrium.

PROOF : See Appendix. ■

A complete-market economy is GEI irreducible if preferences are monotonic and the initial endowments are positive for all agents.

Proposition 3.3: Suppose that Assumptions (A1)–(A5) hold and the asset markets are complete. Then economy $\mathcal{E}$ has equilibrium if and only if it is GEI irreducible.

PROOF : See Appendix. ■

This result implies that market-completing financial innovation always ensures that all agents are free from state-contingent poverty. The equivalence between GEI irreducibility and the existence of equilibrium for complete-market economies is no longer carried over to the case with incomplete-market economies because market incompleteness may deter income transfers necessary to finance cheaper consumptions in equilibrium.

We provide two examples to see how market incompleteness affect the GEI irreducibility of the economy. The first example is the economy mentioned in the introduction that has no-trade equilibrium and is not GEI irreducible because severe market incompleteness forces an agent into SC poverty. Robust survival trivially holds in this example.

The second example is subtler than the first one in that the economy is not GEI irreducible although agents have preferences with all the desired properties such as monotonicity and Inada conditions, and moreover possess cheaper consumptions in equilibrium. Robust survival holds in this example and thus has more latitude than GEI irreducibility in explaining equilibrium with the cheaper consumption property.

Example 3.1 : We consider a one-good, two-agent, three-state economy with two assets where $e_1 \gg 0$, $e_2 = (0,0,a)$ with $a > 0$, and

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Each agent has monotonic preferences. Assumption (A5) holds trivially here because $e_1 \gg 0$. The asset structure does not allow agent 2 to transfer positive income from the third state to the poor states (states 1 and 2). This implies agent 2 is in SC poverty at each $q \in Q$.[8] The no-trade choice is optimal to agent 2 and thus, the economy has no-trade equilibrium. Assumption (A6) trivially holds because $e_1 \gg 0$ and the optimal choice of agent 2 is no-trade at each $q \in Q$. However, the economy is not GEI irreducible because agent 2 is in SC poverty. To show this, let x be an allocation in A_X such that $x_i - e_i = \langle R \rangle$ for each $i = 1,2$. Let z_1 be a point in $\mathbb{R}^3$ with $x_1 + z_1 \succ_1 x_1$ that satisfies $z_1 = R \cdot \eta_1$ for some $\eta_1 = (\eta_{11}, \eta_{12})$ in $\mathbb{R}^2$. Since the preferences are monotonic, the condition $x_1 + z_1 \succ_1 x_1$ implies that one of η_{11} and η_{12} is positive. Recalling that $e_2(1) = e_2(2) = 0$, we have $e_2 - z_1 \notin \mathbb{R}^{\ell}_{+}$. Thus, the economy is not GEI irreducible. □

[8]The consumption good is used as a numeraire in each state.

Example 3.2: We consider a one-good two-agent exchange economy where $S = 3, J = 2$, and the payoff matrix is given by

$$R = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

The agents have utility functions and endowments as follows:

$$u_1(x_1) = 2\sqrt{a_1} + \sqrt{b_1} + \sqrt{c_1}, \quad e_1 = (3,5,4),$$
$$u_2(x_2) = \sqrt{a_2} + \sqrt{6b_2} + \sqrt{c_2}, \quad e_2 = (6,0,6),$$

where $x_i = (a_i, b_i, c_i)$ for each $i = 1,2$. Each u_i is strictly monotonic and convex, and smooth. In particular, the function $\sqrt{a}$ from $\mathbb{R}_+$ to $\mathbb{R}$ satisfies Inada conditions. Thus, the utility functions in the example satisfy all the desired properties assumed in the literature. Agent 1 is endowed with a positive amount of the good in each state while agent 2 is endowed with nothing in the second state. The economy has an equilibrium (q^*, x^*, θ^*) where $q^* = (1.01346, 1)$ and

$$(x_1^*, \theta_1^*) = ((5.4706, 4.9667, 1.4629), (2.4706, -2.5038)),$$
$$(x_2^*, \theta_2^*) = ((3.5294, 0.0333, 8.5371), (-2.4706, 2.5038)).$$

The GEI irreducibility condition fails in the example, however, even though every agent has cheaper consumptions in equilibrium. To show this, we take an individually rational allocation $(\tilde{x}, \tilde{\theta})$ in A where

$$(\tilde{x}_1, \tilde{\theta}_1) = ((41/12, 59/12, 41/12), (25/60, -1/2)),$$
$$(\tilde{x}_2, \tilde{\theta}_2) = ((67/12, 1/12, 79/12), (-25/60, 1/2)).$$

Let $\mathcal{I}_1 = \{2\}$ and $\mathcal{I}_2 = \{1\}$. We pick $\phi = (\phi_1, \phi_2)$, z_1 and z_2 such that $z_1 = R \cdot \phi$ and $z_2 = R \cdot (-\phi)$. Note that the condition $e_2 + z_2 \in \mathbb{R}^3_+$ implies $\phi_1 \leq 6, \phi_1 + \phi_2 \leq 0$, and $\phi_1 + 2\phi_2 \leq 6$. Under these constraints, the function

$$u_1(\tilde{x}_1 + z_1) = \sqrt{41/12 + \phi_1} + \sqrt{59/12 + \phi_1 + \phi_2} + \sqrt{41/12 + \phi_1 + 2\phi_2}$$

achieves maximum at $\phi = 0$. This implies that there is no $\phi \in \mathbb{R}^2$ such that $\tilde{x}_1 + z_1 = \tilde{x}_1 + R \cdot \phi \succ_1 \tilde{x}_1$. Therefore, the economy fails to be GEI irreducible.

Now we show that Assumption (A6) holds here. Each $q = (q_1, 1)$ is in $q \in Q$ for all $q_1 > 1/2$. We set $p = (1,1,1)$. If $q_1 \neq 1$, it is trivial to check the validity of Assumption (A6). If $\bar{q} = (1,1)$, it holds that $\mathcal{B}_1(p, \bar{q}) \neq \emptyset$ but $\mathcal{B}_2(p, \bar{q}) = \emptyset$. Agent 1 has the optimal portfolio choice $\bar{\theta}_1 = (2.6, -2.6)$ and thus the optimal consumption choice $\bar{x}_1 = (5.6, 5, 1.4)$ at $\bar{q}$. In this case, the market-clearing assignment for agent 1 is $\bar{\theta}_2 = (-2.6, 2.6)$ and $\bar{x}_2 = (3.4, 0, 8.6)$. On the other hand, the optimal consumption choice of agent 2 at $\bar{q}$ is the endowment e_2. This implies that $\mathcal{I}(p, \bar{q}, \bar{x}) = \{1\}$. We choose $y_2 = e_2$ and $\eta_2 = (0,0)$. Let q^n be a sequence in Q convergent to $\bar{q}$. We set $y_2^n = e_2$ and $\eta_2^n = (0,0)$ for all n. Clearly, $(y_2^n, \eta_2^n) \in \mathcal{B}(p, q^n)$ for all n and $u_2(y_2) = u_2(e_2) > u_2(\bar{x}_2)$. Consequently, Assumption (A6) holds. □

4. EXISTENCE OF EQUILIBRIUM

To verify the existence of equilibrium, we take a different route from Gottardi and Hens (1996) that perturb the initial endowments to build a sequence of economies with the strong survival condition. Instead, the preference orderings and the consumption sets are perturbed here in a peculiar way to build a sequence of economies that satisfy the strong survival condition. This approach may have a merit in dealing with the discontinuity problem of the budget sets.[9] For a sequence $\{\varepsilon_n\}$ converging to 0 in $\mathbb{R}^\ell_{++}$, we define a preference ordering P_i^n on $\mathbb{R}^\ell_+ - \{\varepsilon_n\}$ such that, for each $x_i \in \mathbb{R}^\ell_+ - \{\varepsilon_n\}$,

$$P_i^n(x_i) = P_i(x_i + \varepsilon_n).$$

Let $\mathcal{E}^n$ denote the economy $\langle(\mathbb{R}^\ell_+ - \{\varepsilon_n\}, P_i^n, e_i, \mathbb{R}^J)_{i\in\mathcal{I}}, R\rangle$. For each $(p,q) \in \mathbb{R}^\ell_+ \times c\ell Q$ and each n, let $\mathcal{B}_i^n(p,q)$ denote the open budget set $\mathcal{B}_i(p,q)$ of agent i with the consumption set $\mathbb{R}^\ell_+$ replaced by $\mathbb{R}^\ell_+ - \{\varepsilon_n\}$.

$$\mathcal{B}_i^n(p,q) = \{(x_i, \theta_i) \in (\mathbb{R}^\ell_+ - \{\varepsilon_n\}) \times \mathbb{R}^J : p \,\square\, (x_i - e_i) \ll W_i(q) \cdot \theta_i\}$$

Noting that $e_i > 0$ and $e_i \in \mathbb{R}^\ell_+ - \{\varepsilon_n\}$ for each $i \in \mathcal{I}$ and n, each $\mathcal{E}^n$ satisfies the strong survival condition, i.e., e_i is in the interior of the consumption set for $\mathcal{E}^n$.

We introduce the sets of normalized prices:

$$D_t = \Delta_0 \times \Delta_1 \text{ where } \Delta_0 = \{q \in Q : \|q\| = 1\},$$
$$\Delta_1 = \textstyle\prod_{s\in\mathcal{S}} \Delta_s \text{ with } \Delta_s = \left\{p(s) \in \mathbb{R}^L_{++} : \|p(s)\| = 1\right\}^{10}$$

[9] In the literature, the continuity of the budget set is defined in the price domain alone. The discontinuity problem can be sometimes obviated by augmenting the domain of the budget correspondence with other variables. This is the reason why consumptions are allowed to be perturbed into the negative quantities in Assumption (A6). For example, consider an economy that coincides with the economy in Example 3.2 except that agent 2 has a utility function $u_2(x_2) = \sqrt{a_2} + \sqrt{3(b_2+2)} + \sqrt{c_2}$ with $e_2 = (2,0,1)$. The economy satisfies robust survival and has an equilibrium. However, the budget set is not lower hemicontinuous at $q = (1,1)$. For details about the lower hemicontinuity, see Hildenbrand (1974).

Competitive equilibrium is defined for $\mathcal{E}^n$ as in Definition 2.1. By invoking the result of Werner (1989) in $\{\mathcal{E}^n\}$, we see that each $\mathcal{E}^n$ has a competitive equilibrium.

Proposition 4.1: Under Assumptions (A1)–(A3), each $\mathcal{E}^n$ has a competitive equilibrium in $\Delta \times (\mathbb{R}^\ell_+ - \{\varepsilon_n\})^I \times \mathbb{R}^{JI}$.

The following shows that the limit of a sequence of equilibria for $\{\mathcal{E}^n\}$ becomes an equilibrium of $\mathcal{E}$ under Assumptions (A5) and (A6).

Theorem 4.1: Under Assumptions (A1)–(A6), economy $\mathcal{E}$ has a competitive equilibrium $(p^*, q^*, x^*, \theta^*) \in \Delta \times A$.

PROOF : See Appendix. ∎

By Proposition 3.1, GEI irreducibility implies (A6). Thus, Theorem 4.1 encompasses the existence result of Gottardi and Hens (1996) as a special case.

When asset markets are complete, the following corollary comes from Propositions 3.1 and 3.3, and Theorem 4.1.

Corollary 4.1: Suppose that Assumptions (A1)–(A5) hold. When asset markets are complete, the following are equivalent:

(1) Economy $\mathcal{E}$ has a competitive equilibrium.

(2) Economy $\mathcal{E}$ is GEI reducible.

(3) Economy $\mathcal{E}$ satisfies robust survival.

PROOF : Proposition 3.3 ensures the equivalence between (1) and (2). By Proposition 3.1, (2) implies (3). The necessity of robust survival for the GEI irreducibility is immediate from Theorem 4.1. ∎

5. IMPLICATIONS OF SURVIVAL ASSUMPTIONS TO FINANCIAL INNOVATION

We consider financial innovation that changes the payoff matrix R to a new payoff matrix R_F of dimension $S \times J_F$ where J_F is an integer greater than J. It is assumed that R is a submatrix of R_F and the J_F rows in R_F are linearly independent. Since the new assets strictly enlarge risk-sharing opportunities, it holds that $\langle R \rangle \subset \langle R_F \rangle$ and $\langle R_F \rangle \setminus \langle R \rangle \neq \emptyset$. Let $\mathcal{E}_F$ denote the post-innovation economy, i.e., $\mathcal{E}_F$ coincides with $\mathcal{E}$ except that R_F replaces R. As illustrated in Hart (1975) and Elul (1995), the expansion of trading opportunities may not necessarily lead to overall welfare improvement for all agents. Here, we focus on a special case where an increase in the asset span can improve the welfare of agents who experience state-contingent poverty.

Definition 5.1: The financial innovation of shifting R to R_F is *benevolent to the state-contingent poor* (SC poor, for short) in economy $\mathcal{E}$ if economy $\mathcal{E}_F$ has the cheaper consumption property in every equilibrium.

Suppose that the pre-innovation economy has equilibrium where some agents are in SC poverty. Financial innovation will be benevolent to the SC poor if the post-innovation equilibria satisfy the cheaper consumption property. By Proposition 3.2, we see that the cheaper consumption property holds in $\mathcal{E}_F$ if it is GEI irreducible and R_F allows a portfolio to generate a positive payoff in each state. Consequently, the following result is obtained as a corollary of Proposition 3.2.

Corollary 5.1: Suppose that (i) Assumptions (A1)–(A5) hold, (ii) $R \cdot v^\circ \gg 0$ for some $v^\circ \in \mathbb{R}^J$, (iii) the economy $\mathcal{E}$ satisfies robust survival, and (iv) the new economy $\mathcal{E}_F$ is GEI irreducible. Then financial innovation is benevolent to the SC poor.

As illustrated below, the market system may break down in the post innovation stage if financial innovation is not benevolent to the SC poor. When $\mathcal{E}$ is not GEI irreducible, SC poverty may arise in equilibrium of $\mathcal{E}$. If the post-innovation economy $\mathcal{E}_F$ is GEI irreducible, Proposition 3.2 implies that SC poverty disappears in new equilibrium and thus financial innovation is benevolent to the SC poor.

Remark: Definition 5.1 focuses on the capability of financial innovation to improve the welfare of the SC poor. It is possible that a benevolent financial innovation can have negative effects on the welfare of other agents, making them worse off. In this case, financial innovation may be regarded as a redistributive mechanism. Definition 5.1 can be extended to the case where every agent is free from SC poverty, but some agents are so close to SC poverty that they are considered relatively poor.

For instance, suppose that some agent i has an optimal consumption x_i where the expenditure on $x_i(s)$ is close to zero for some state s. Let $\tilde{X}_i$ denote the closure of the convex hull generated by $P_i(x_i)$ and e_i. We consider an economy $\mathcal{E}_i$ which is identical to $\mathcal{E}$ except that X_i is replaced by $\tilde{X}_i$ and P_i is replaced by the restriction of P_i to $\tilde{X}_i$. We can modify Definition 5.1 in terms of relative poverty and apply the same methodology developed above to the economy $\mathcal{E}_i$ to check that relative poverty is eliminated in the post-innovation equilibrium.

Financial innovation is benevolent to the SC poor if it makes $\mathcal{E}_F$ GEI irreducible. Two examples are discussed below to illustrate how the two survival conditions, robust survival and GEI irreducibility, can be coupled to analyze the effect of financial innovation on equilibrium and welfare. The first example demonstrates that comparative statics on the welfare change of the SC poor induced by financial innovation can be conducted by subjecting the pre-innovation economy to the robust survival test and then applying the irreducibility test to the post-innovation economy. The example starts with a two-asset, four-state economy that satisfies robust survival and is not GEI irreducible. The failure of GEI irreducibility prevents one of the agents from possessing cheaper consumptions in a state. The economy becomes GEI irreducible after an European put option is introduced through financial innovation. The cheaper consumption property holds in post-innovation equilibrium, implying that the financial innovation is benevolent to the SC poor.

The second example shows that when financial innovation occurs in the presence of the SC poor, increased risk-sharing opportunities can lead to market failure. In this case, it becomes a curse to the invisible hand.

Example 5.1: The following economy is an augmented version of the economy in Example 2 of Gottardi and Hens (1996).[11] In particular, the fourth state is added to the uncertainty to see the effect of financial innovation on the SC poor. The payoff matrix is given by

$$R = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

The two agents have the same utility function $\sqrt{a} + \sqrt{b} + \sqrt{c} + \sqrt{d}$ and are endowed with $e_1 = (1,1,1,1)$ and $e_2 = (1,0,1,1)$, respectively. The economy has a unique equilibrium (q^*, x^*, θ^*) where $q^* = (1,1)$, $x_i^* = e_i$ and $\theta_i^* = (0,0)$ for each $i = 1,2$. The second agent has no cheaper consumption in the second state in equilibrium and thus falls in SC poverty. Since $R \cdot v^\circ \gg 0$ with $v^\circ = (1,1)$, by Proposition 3.2, the economy is not GEI irreducible. However, Assumption (A6) holds trivially because no trade is the optimal choice for both agents.

Now we introduce an European put option on the second asset with strike price \$1 that expires in the second period. The financial innovation leads to the

[11] A typo in the payoff matrix in Gottardi and Hens (1996) is fixed here.

augmented payoff matrix.

$$R_F = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

We claim that the post-innovation economy is GEI irreducible. To see this, let x be any allocation in A_X and $\mathcal{I}_1$ and $\mathcal{I}_2$ denote a nontrivial partition of $\mathcal{I}$. Consider the case with $\mathcal{I}_1 = \{2\}$ and $\mathcal{I}_2 = \{1\}$. We set $\phi_1 = (0,0,1)$ and $\phi_2 = (0,0,-1)$. Then $z_1 := R_F \cdot \phi_1 = (1,0,0,0)$ and $z_2 := R_F \cdot \phi_2 = (-1,0,0,0)$. Since preferences are strictly monotonic, it is clear that $x_1 + z_1 \in P_1(x_1)$ and $e_2 + z_2 \in \mathbb{R}^\ell_+$. When $\mathcal{I}_1 = \{1\}$ and $\mathcal{I}_2 = \{2\}$, we set $\phi_1 = (0,0,-1)$ and $\phi_2 = (0,0,1)$. Then $z_1 := R_F \cdot \phi_1 = (-1,0,0,0)$ and $z_2 := R_F \cdot \phi_2 = (1,0,0,0)$. It holds that $x_2 + z_2 \in P_2(x_2)$ and $e_1 + z_1 \in \mathbb{R}^\ell_+$. Thus, the post-innovation economy is GEI irreducible.

Since $R_F \cdot v^\circ \gg 0$ where $v^\circ = (1,1,1)$, by Proposition 3.2, the GEI irreducibility of the post-innovation economy implies that every agent has cheaper consumptions in equilibrium. Since asset trading occurs, every agent gets better off in the new equilibrium. In particular, financial innovation improves the welfare of the SC poor. Specifically, the economy has an equilibrium where

$$\begin{aligned} q^{**} &= (3.5670, 3.5670, 1), \\ (x_1^{**}, \theta_1^{**}) &= ((1.2835, 0.7791, 1.2835, 0.7791), (-0.5044, 0.2835, 0.7879)), \\ (x_2^{**}, \theta_2^{**}) &= ((0.7165, 0.2209, 0.7165, 1.2209), (0.5044, -0.2835, -0.7879)). \end{aligned}$$

It yields the following comparative statics on individual welfare.

$$u_1(x_1^{**}) = 4.03 > u_1(x_1^*) = 4 \quad \text{and} \quad u_2(x_2^{**}) = 3.267 > u_2(x_2^*) = 3.$$

Agent 2 gets better off because financial innovation provides an opportunity to gains access to cheaper consumptions in the new equilibrium. □

Example 5.2: This is the example discussed in the introduction that is based on Example 2 of Gottardi and Hens (1996). We start with the single-asset economy with the payoff matrix given by

$$R = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

The two agents have the same utility function $\sqrt{a}+\sqrt{b}+\sqrt{c}$ and are endowed with $e_1 = (1,1,1)$ and $e_2 = (1,0,\alpha)$ with $0 \leq \alpha < 1/2$, respectively. The economy has a no-trade equilibrium. By the same arguments made in Example 3.1, it satisfies robust survival but is not GEI irreducible. Suppose that financial innovation augments the payoff matrix to the one in Example 2 of Gottardi and Hens (1996):

$$R_F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Gottardi and Hens (1996) show that the post-innovation economy is not GEI irreducible and has no equilibrium.[12] In this case, financial innovation is a curse to the market system. □

6. CONCLUSION

The existing literature have paid little attention to the existence of GEI equilibrium in which some agents experience state-contingent poverty. As illustrated in Examples 3.1 and 5.1, the state-contingent poverty can occur in the GEI model where agents have the initial endowment in the boundary of the consumption set. This case reflects the realism that individuals in the real world are not endowed with all available goods and services, highlighting the gray area in general equilibrium analysis. This paper has introduced the concept of robust survival, which is weaker than GEI irreducibility but can explain GEI equilibria where agents are unable to afford cheaper consumptions in certain states. The examples show that state-contingent poverty is more likely to occur in economies with a higher degree of market incompleteness. Robust survival provides insights into the problem of poverty in the GEI model.

The combination of robust survival and the GEI irreducibility condition serves as a criterion for evaluating the impact of financial innovation on the welfare of the SC poor. Example 5.1 illustrates that poverty can arise in a GEI economy that satisfies robust survival but is not GEI irreducible. However, the introduction of a put option on the second asset makes the economy irreducible in this example, thereby eliminating poverty in the post-innovation equilibrium. When the lower bound of the consumption set defines the poverty threshold, the cheaper consumption property becomes a normative goal. Financial innovation can help achieve this goal by making the economy irreducible.

[12] The economy does not satisfy robust survival either.

The paper unveils the potential of paired survival conditions to provide a conceptual foundation for addressing the poverty problem and analyzing the effects of policy options on the SC poor in the GEI model.

REFERENCES

Elul, R. n (1995). "Welfare Effects of Financial Innovation in Incomplete Markets Economies with Several Consumption," *Journal of Economic Theory*, 65, 43–78.

Gottardi, P., and T. Hens (1996). "The Survival Assumption and Existence of Competitive Equilibria When Asset Markets are Incomplete," *Journal of Economic Theory*, 71, 313–323.

Hahn, G., and D. C. Won (2012). "Survival, Arbitrage, and Equilibrium with Financial Derivatives in Constrained Asset Markets," *Seoul Journal of Economics*, 25(4), 442–461.

Hart, O. D. (1975). "On the Optimality of Equilibrium when the Market Structure is Incomplete," *Journal of Economic Theory*, 11, 418–443.

McKenzie, L. (1959). "On the Existence of General Equilibrium for a Competitive Market," *Econometrica*, 27, 54–71.

McKenzie, L. (1981). "The Classical Theorem on Existence of Competitive Equilibrium," *Econometrica*, 49, 819–841.

Werner, J. (1989). "Equilibrium with Incomplete Markets without Ordered Preferences," *Journal of Economic Theory*, 49, 379–382.

Hildenbrand, W. (1974). "Core and Equilibria of a Large Economy," *Princeton University Press*.

APPENDIX

A.1. Proof of Proposition 3.1

PROOF : Let (p,q,x,θ) be a point in $\mathbb{R}^{\ell}_{++} \times Q \times A$ with $\mathcal{I} \setminus \mathcal{I}(p,q,x) \neq \emptyset$ that satisfies $p \square_1 (x_i - e_i) = R \cdot \theta_i$, and (p^n, q^n) be a sequence in $\mathbb{R}^{\ell}_{++} \times Q$ with $(p^n, q^n) \to (p,q)$. By Assumption (A5), there exists an agent $h \in \mathcal{I}$ with $\zeta_h \in \mathbb{R}^J$ that satisfies $q \cdot \zeta_h < 0$ and $p \square_1 e_h + R \cdot \zeta_h \geq 0$. The following two cases are conceivable:

- (CASE 1): (x_h, θ_h) is not optimal at (p,q), i.e., $h \in \mathcal{I} \setminus \mathcal{I}(p,q,x)$.
- (CASE 2): (x_h, θ_h) is optimal at (p,q), i.e., $h \in \mathcal{I}(p,q,x)$.

(Case 1) Suppose that $h \in \mathcal{I} \setminus \mathcal{I}(p,q,x)$. Noting that $q \cdot \theta_i = 0$ for each $i \in \mathcal{I}(p,q,x)$ and $\sum_{i \in \mathcal{I}} \theta_i = 0$, we have $\sum_{i \in \mathcal{I} \setminus \mathcal{I}(p,q,x)} q \cdot \theta_i = 0$. This implies that either [SUBCASE $(1a)$] $q \cdot \theta_i = 0$ for all $i \in \mathcal{I} \setminus \mathcal{I}(p,q,x)$ or [SUBCASE $(1b)$] $q \cdot \theta_i < 0$ for some $i \in \mathcal{I} \setminus \mathcal{I}(p,q,x)$.

[SUBCASE $(1a)$] In particular, the pair (x_h, θ_h) satisfies $q \cdot \theta_h = 0$ and $p \square_1 (x_h - e_h) = R \cdot \theta_h$. Since $h \in \mathcal{I} \setminus \mathcal{I}(p,q,x)$, there exists $(y'_h, \eta'_h) \in c\ell \mathcal{B}_h(p,q)$ with $y'_h \succ_h x_h$. We set $x'_h = x_h/2$ and $\theta'_h = \theta_h/2 + \zeta_h/2$. Then we obtain $q \cdot \theta'_h < 0$ and $p \square_1 x'_h \leq p \square_1 e_h + R \cdot \theta'_h$. Let β be a point in $(0,1)$ sufficiently close to 1 such that $\beta y'_h + (1-\beta) x'_h \succ_h x_h$. We set $\tilde{y}_h = \beta y'_h + (1-\beta) x'_h$ and $\tilde{\eta}_h = \beta \eta'_h + (1-\beta)\theta'_h$. Then $(\tilde{y}_h, \tilde{\eta}_h)$ satisfies $\tilde{y}_h \succ x_h$, $q \cdot \tilde{\eta}_h < 0$, and $p \square_1 \tilde{y}_h \leq p \square_1 e_h + R \cdot \tilde{\eta}_h$.

Let $\mathcal{S}_0$ denote the set $\{s \in \mathcal{S} : e_h(s) = 0\}$. Then $\mathcal{S}$ is divided into the two sets $\mathcal{S}_0$ and $\mathcal{S} \setminus \mathcal{S}_0$. For each $s \in \mathcal{S}_0$, we have $0 \leq p(s) \cdot \tilde{y}_h(s) \leq r(s) \cdot \tilde{\eta}_h$. If $p(s) \cdot \tilde{y}_h(s) > 0$, then it holds that $0 < p(s) \cdot \tilde{y}_h(s) \leq r(s) \cdot \tilde{\eta}_h$. In this case, we choose a sequence $\{\tilde{y}^n_h(s)\}$ in $\mathbb{R}^L_+$ such that $\tilde{y}^n_h(s) \to \tilde{y}_h(s)$ and $p^n(s) \cdot \tilde{y}^n_h(s) \leq r(s) \cdot \tilde{\eta}_h$. If $p(s) \cdot \tilde{y}_h(s) = 0$, it is clear that $\tilde{y}_h(s) = 0$ (since $p \gg 0$) and $r(s) \cdot \tilde{\eta}_h \geq 0$. By setting $\tilde{y}^n_h(s) = 0$ for all n, we have $\tilde{y}^n_h(s) \to \tilde{y}_h(s)$ and $p^n(s) \cdot \tilde{y}^n_h(s) \leq r(s) \cdot \tilde{\eta}_h$. Since $e_h(s) = 0$ for all $s \in \mathcal{S}_0$, we see that $p^n(s) \cdot \tilde{y}^n_h(s) \leq p^n(s) \cdot e_h(s) + r(s) \cdot \tilde{\eta}_h$. Thus, for any $\alpha \in (0,1)$ it holds that $p^n(s) \cdot (\alpha \tilde{y}^n_h(s)) \leq p^n(s) \cdot e_h(s) + r(s) \cdot (\alpha \tilde{\eta}_h)$.

We turn to the case that $s \in \mathcal{S} \setminus \mathcal{S}_0$. Since $p \gg 0$, we have $p(s) \cdot e_h(s) > 0$. Then for any $\alpha \in (0,1)$, it holds that $p(s) \cdot (\alpha \tilde{y}_h(s)) < p(s) \cdot e_h(s) + r(s) \cdot (\alpha \tilde{\eta}_h)$. We set $\tilde{y}^n_h(s) = \tilde{y}_h(s)$ for all n. Then for sufficiently large n, we have $p^n(s) \cdot (\alpha \tilde{y}^n_h(s)) < p^n(s) \cdot e_h(s) + r(s) \cdot (\alpha \tilde{\eta}_h)$.

Until now we have shown that $q^n \cdot \alpha \tilde{\eta}_h < 0$ and $p^n \square_1 (\alpha \tilde{y}^n_h - e_h) \leq R \cdot (\alpha \tilde{\eta}_h)$ for sufficiently large n. Now let α be a number in $(0,1)$ sufficiently close to 1 such that $\alpha \tilde{y}_h \succ_h x_h$. We set $y_h = \alpha \tilde{y}_h$, $\eta_h = \alpha \tilde{\eta}_h$, and $y^n_h(s) = \alpha \tilde{y}^n_h(s)$ and $\eta^n_h =$

$\alpha\tilde{\eta}_h$ for all n and $s \in \mathcal{S}$. Then it follows that $(y_h^n, \eta_h^n) \to (y_h, \eta_h)$ and $(y_h^n, \eta_h^n) \in c\ell\mathcal{B}_h(p^n, q^n)$ for sufficiently large n.

[SUBCASE $(1b)$] The pair (x_i, θ_i) satisfies $q \cdot \theta_i < 0$ and $p \,\square_1 (x_i - e_i) \le R \cdot \theta_i$. Since $i \in \mathcal{I} \setminus \mathcal{I}(p,q,x)$, there exists $(y_i', \eta_i') \in c\ell\mathcal{B}_i(p,q)$ with $y_i' \succ_i x_i$. By the same arguments made in Subcase $(1a)$), there exists $(y_i, \eta_i) \in c\ell\mathcal{B}_i(p,q)$ with $y_i \succ_i x_i$ and a sequence $\{(y_i^n, \eta_i^n)\}$ that satisfies $(y_i^n, \eta_i^n) \to (y_i, \eta_i)$ and $(y_i^n, \eta_i^n) \in c\ell\mathcal{B}_i(p^n, q^n)$ for all n. □

(Case 2) Consider the case that $h \in \mathcal{I}(p,q,x)$. To match the notation used in (A7), we set $\mathcal{I}_1 = \mathcal{I} \setminus \mathcal{I}(p,q,x)$ and $\mathcal{I}_2 = \mathcal{I}(p,q,x)$. Since $\mathcal{I}_1 \neq \emptyset$ and $\mathcal{I}_2 \neq \emptyset$, $\{\mathcal{I}_1, \mathcal{I}_2\}$ is a nontrivial partition. By (A7) we can take $z_i \in \mathbb{R}^\ell$ and $\phi_i \in \mathbb{R}^J$ for each $i \in \mathcal{I}$ such that $p \,\square_1 z_i = R \cdot \phi_i$ for each $i \in \mathcal{I}$, $x_i + z_i \succ_i x_i$ for each $i \in \mathcal{I}_2$, $e_i + z_i \in \mathbb{R}^\ell_+, \forall i \in \mathcal{I}_1$, $\sum_{i \in \mathcal{I}} z_i = 0$, and $\sum_{i \in \mathcal{I}} \phi_i = 0$. Since $p \,\square_1 (x_i - e_i) = R \cdot \theta_i$ for each $i \in \mathcal{I}$, it holds that for all $i \in \mathcal{I}_2$, $p \,\square_1 (x_i + z_i - e_i) = R \cdot (\theta_i + \phi_i)$. Thus, $q \cdot (\theta_i + \phi_i) > 0$ for each $i \in \mathcal{I}_2$. Summing it over all the agents in $\mathcal{I}_2$, we obtain $q \cdot \sum_{i \in \mathcal{I}_2} (\theta_i + \phi_i) > 0$. Noting that $q \cdot \theta_i \le 0$ for each $i \in \mathcal{I}_2$, it gives $q \cdot \sum_{i \in \mathcal{I}_2} \phi_i > 0$. Recalling that $\sum_{i \in \mathcal{I}} \phi_i = 0$, we obtain $q \cdot \sum_{i \in \mathcal{I}_1} \phi_i < 0$. Therefore, there exists an agent $i_0 \in \mathcal{I}_1$ such that $q \cdot \phi_{i_0} < 0$. We set $x_{i_0}' = e_{i_0} + z_{i_0}$. It holds that $x_{i_0}' \in \mathbb{R}^\ell_+$ and $p \,\square_1 (x_{i_0}' - e_{i_0}) = p \,\square_1 z_{i_0} = R \cdot \phi_{i_0}$. By a similar argument made in Case 1, we can find (y_{i_0}, η_{i_0}) in $c\ell\mathcal{B}_{i_0}(p,q)$ with $y_{i_0} \succ_{i_0} x_{i_0}$ and $\{(y_{i_0}^n, \eta_{i_0}^n)\}$ in $\mathbb{R}^\ell_+ \times \mathbb{R}^J$ that satisfies $(y_{i_0}^n, \eta_{i_0}^n) \to (y_{i_0}, \eta_{i_0})$ and $(y_{i_0}^n, \eta_{i_0}^n) \in c\ell\mathcal{B}_{i_0}(p^n, q^n)$ for each n. Thus, Assumption (A6$'$) holds. ■

A.2. Proof of Proposition 3.2

PROOF : Let (p,q,x,θ) denote an equilibrium of the economy that is GEI irreducible. We set $\mathcal{I}_2 = \{i \in \mathcal{I} : \mathcal{B}_i(p,q) \neq \emptyset\}$ and $\mathcal{I}_1 = \mathcal{I} \setminus \mathcal{I}_2$. Since there exists $v^\circ \in \mathbb{R}^J$ with $R \cdot v^\circ \gg 0$, Assumption (A5) implies $\mathcal{I}_2 \neq \emptyset$. Suppose that $\mathcal{I}_1 \neq \emptyset$. Then the same arguments made in Case 2 of the proof of Proposition 3.1 leads to a contradiction. Thus, we conclude that $\mathcal{I}_1 = \emptyset$. ■

A.3. Proof of Proposition 3.3

PROOF : Sufficiency is proved in Gottardi and Hens (1996). To prove necessity, let $\{\mathcal{I}_1, \mathcal{I}_2\}$ be any nontrivial partition of $\mathcal{I}$, p a price in $\mathbb{R}^\ell_{++}$, and (x,θ) an allocation in A that satisfies $p \,\square_1 (x_i - e_i) = R \cdot \theta_i, \forall i \in \mathcal{I}$. Let us take an agent h in $\mathcal{I}_1$. Since markets are complete, there exists ϕ_h in $\mathbb{R}^J$ such that $p \,\square_1 e_h = R \cdot \phi_h$. We set $z_h = -e_h$, $z_i = 0$ for each $i \in \mathcal{I}_1 \setminus \{h\}$, and $z_i = e_h / N(\mathcal{I}_2)$ for each $i \in \mathcal{I}_2$, where $N(\mathcal{I}_2)$ indicates the number of agents in $\mathcal{I}_2$. By construction, it holds that $e_i + z_i \in \mathbb{R}^\ell_+$ for all $i \in \mathcal{I}_1$, $p \,\square_1 z_h = R \cdot (-\phi_h)$, $p \,\square_1 z_i = 0$ for each $i \in \mathcal{I}_1 \setminus \{h\}$, $p \,\square_1 z_i = R \cdot (\phi_h / N(\mathcal{I}_2))$ for each $i \in \mathcal{I}_2$, and $\sum_{i \in \mathcal{I}} z_i = 0$. Since $e_h > 0$ and preferences are strictly monotonic, we see that, for each $i \in \mathcal{I}_2$, $x_i + z_i \in P_i(x_i)$. Therefore, the economy $\mathcal{E}$ is GEI irreducible. ■

A.4. Proof of Theorem 4.1

PROOF : By Proposition 4.1, $\mathcal{E}^n$ has a competitive equilibrium $(p^n, q^n, x^n, \theta^n) \in \Delta \times (\mathbb{R}^\ell_+ - \{\varepsilon_n\})^I \times \mathbb{R}^{JI}$. Recalling that for sufficiently large n, $-1_\ell \le x_i^n \le \sum_{i \in \mathcal{I}} e_i$ where $1_\ell = (1, 1, \ldots, 1) \in \mathbb{R}^\ell$, we see that sequence $\{x^n\}$ is bounded. Since R has full rank, sequence $\{\theta^n\}$ is bounded as well. Without loss of generality, we can assume that sequence $\{(p^n, q^n, x^n, \theta^n)\}$ converges to a point $(p^*, q^*, x^*, \theta^*) \in c\ell\Delta \times A$. Thus it is observed that $p \,\square\, (x_i^* - e_i) = W(q^*) \cdot \theta^*$ for every $i \in \mathcal{I}$ and $(x^*, \theta^*) \in A$. We need to check that $(p^*, q^*) \in \Delta$ and (x_i^*, θ_i^*) is optimal at (p^*, q^*) for each i.

CLAIM 1: $p^* \in \Delta_1$.

PROOF : Suppose otherwise. Then $p^*(s) \in c\ell\Delta_s \setminus \Delta_s$ for some $s \in S$. We can pick $\delta \in \mathbb{R}^\ell_+$ such that $\delta(s) > 0$ with $p^*(s) \cdot \delta(s) = 0$, and $\delta(s') = 0$ if $s' \ne s$. Since $\sum_{i \in \mathcal{I}} e_i \gg 0$, there exists an agent $i \in \mathcal{I}$ such that $p^*(s) \cdot e_i(s) > 0$. Then, for each $\alpha \in (0,1)$, we have $p^*(s) \cdot [\alpha(x_i^*(s) + \delta(s))] < p^*(s) \cdot e_i(s) + r(s) \cdot (\alpha\theta_i^*)$ and $p^*(s') \cdot [\alpha(x_i^*(s') + \delta(s'))] \le p^*(s') \cdot e_i(s') + r(s') \cdot (\alpha\theta_i^*)$ for all $s' \ne s$. On the other hand, $\alpha(x_i^* + \delta) \succ_i x_i^*$ for some $\alpha \in (0,1)$. Since $x_i^n \in \mathbb{R}^\ell_+ - \{\varepsilon_n\}$ and $x_i^n \to x_i^* \in \mathbb{R}^\ell_+$, this implies that for sufficiently large n, $\alpha(x_i^n + \delta) + \varepsilon_n \succ_i x_i^n + \varepsilon_n$ or $\alpha(x_i^n + \delta) \in P_i^n(x_i^n)$. It also holds that for sufficiently large n, $p^n(s) \cdot [\alpha(x_i^n(s) + \delta(s))] < p^n(s) \cdot e_i(s) + r(s) \cdot (\alpha\theta_i^n)$ and $p^n(s') \cdot [\alpha(x_i^n(s') + \delta(s'))] \le p^n(s') \cdot e_i(s') + r(s') \cdot (\alpha\theta_i^n)$ for all $s' \ne s$ (This is because $\delta(s') = 0$ for all $s' \ne s$). It holds that $p^n \,\square_1\, [\alpha(x_i^n + \delta) - e_i] \le R \cdot (\alpha\theta_i^n)$ for sufficiently large n. Since $q^n \cdot (\alpha\theta_i^n) = 0$, it follows that $(\alpha(x_i^n + \delta), \alpha\theta_i^n) \in c\ell\mathcal{B}_i^n(p^n, q^n)$. These results contradict the optimality of (x_i^n, θ_i^n) in $c\ell\mathcal{B}_i^n(p^n, q^n)$. Hence, we have $p^* \in \Delta_1$. □

CLAIM 2: $q^* \in \Delta_0$.

PROOF : Suppose otherwise. Then $q^* \in c\ell\Delta_0 \setminus \Delta_0$. By Assumption (A5), we can choose an agent $i \in \mathcal{I}$ with $\zeta_i \in \mathbb{R}^J$ which satisfies $q^* \cdot \zeta_i < 0$ and $p^* \,\square_1\, e_i + R \cdot \zeta_i \ge 0$. Since $q^* \in c\ell Q \setminus Q$, there exists $v_i \in \mathbb{R}^J$ such that $q^* \cdot v_i \le 0$ and $R \cdot v_i > 0$. We take a point $\delta \in \mathbb{R}^\ell_+ \setminus \{0\}$ such that $x_i^* + \delta \in P_i(x_i^*)$ and $0 < p^* \,\square_1\, \delta < R \cdot v_i$. For every $\alpha \in (0,1)$ sufficiently close to 1, it holds that $\alpha x_i^* + \delta \in P_i(x_i^*)$, $q^* \cdot [\alpha\theta_i^* + (1-\alpha)\zeta_i + v_i] < 0$ and $p^* \,\square_1\, (\alpha x_i^* + \delta - e_i) < R \cdot [\alpha\theta_i^* + (1-\alpha)\zeta_i + v_i]$. For each $\alpha_n \in (0,1)$ with $\alpha_n \to 1$, we set $y_i^n = (1-\alpha_n)(-\varepsilon_n) + \alpha_n x_i^* + \delta \in \mathbb{R}^\ell_+ - \{\varepsilon_n\}$. Then for sufficiently large n, we see that $y_i^n \in P_i^n(x_i^n)$, $q^n \cdot [\alpha_n\theta_i^* + (1-\alpha_n)\zeta_i +$

$v_i] < 0$, and

$$p^n \square_1 y_i^n \ll p^n \square_1 e_i + R \cdot [\alpha_n \theta_i^* + (1-\alpha_n)\zeta_i + v_i].$$

Thus $(y_i^n, \alpha_n\theta_i^* + (1-\alpha_n)\zeta_i + v_i) \in c\ell\mathcal{B}_i^n(p^n, q^n)$ and $y_i^n \in P_i^n(x_i^n)$ for sufficiently large n, which contradicts the optimality of (x_i^n, θ_i^n) in $c\ell\mathcal{B}_i^n(p^n, q^n)$. Consequently, we see that q^* belongs to Δ_0. □

CLAIM 3: For every $i \in \mathcal{I}$, it holds that $(P_i(x_i^*) \times \mathbb{R}^J) \cap c\ell\mathcal{B}_i(p^*, q^*) = \emptyset$.

PROOF : Suppose otherwise. Then we have $\mathcal{I}(p^*, q^*, x^*) \neq \mathcal{I}$. Now we show that $\mathcal{I}(p^*, q^*, x^*) \neq \emptyset$. By Assumption (A5), there exists agent i with $\zeta_i \in \mathbb{R}^J$ such that $q \cdot \zeta_i < 0$ and $p^* \square_1 e_i + R \cdot \zeta_i \geq 0$. We claim that $i \in \mathcal{I}(p^*, q^*, x^*)$. Otherwise, there would exist $(y_i, \eta_i) \in c\ell\mathcal{B}_i(p^*, q^*)$ with $y_i \succ_i x_i^*$. By the same argument made in Case 1 of the proof of Proposition 3.1, we can take $(y_i^n, \eta_i^n) \in c\ell\mathcal{B}_i(p^n, q^n) \subset c\ell\mathcal{B}_i^n(p^n, q^n)$ with $y_i^n \succ_i x_i^n$, which contradicts the optimality of (x_i^n, θ_i^n) in $c\ell\mathcal{B}_i^n(p^n, q^n)$.

Then by Assumption (A6), for some $h \in \mathcal{I} \setminus \mathcal{I}(p^*, q^*, x^*)$, there exists $(y_h, \eta_h) \in c\ell\mathcal{B}_h(p^*, q^*)$ with $y_h \succ_h x_h^*$ that admits a sequence $(y_h^n, \eta_h^n) \to (y_h, \eta_h)$ with $(y_h^n, \eta_h^n) \in c\ell\mathcal{B}_h^n(p^n, q^n)$. Since $y_h^n \in P_h^n(x_h^n)$ for sufficiently large n, it contradicts the optimality of (x_h^n, θ_h^n) in $c\ell\mathcal{B}_h^n(p^n, q^n)$. Therefore, we conclude that $\big(P_i(x_i^*) \times \mathbb{R}^J\big) \cap c\ell\mathcal{B}_i(p^*, q^*) = \emptyset$ for every $i \in \mathcal{I}$. □

Hence, $(p^*, q^*, x^*, \theta^*) \in \Delta \times A$ is an equilibrium of the economy. ■