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Election Contests with Endogenous Spending Constraints*

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Abstract We consider an election contest in which two candidates first raise funds and then compete for votes among a continuum of voters by engaging in persuasive efforts. To cover campaign spending, candidates must raise funds by bearing costs beforehand and they compete by allocating persuasive efforts among voters. Each voter is persuaded by campaign effort and votes for the candidate who expends more persuasive effort than the other. We characterize equilibrium strategies—both fund-raising and allocation of persuasive effort strategies. A candidate with a higher value for the vote raises more funds than the other, but the latter competes in the election stage by giving zero persuasive effort with a positive probability to each voter and using the saved money for expending a high level of persuasive effort with the remaining probability. The role of fund-raising costs is also discussed.

Keywords Election contest, endogenous budgets, asymmetric values

JEL Classification D72, H23

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1. INTRODUCTION

In an election competition, each candidate tries to persuade voters by campaign spending to gain more votes. It is traditional for candidates to appeal broadly to all voters via media coverage such as television advertisements. However, the recent development of technology allows them to target their campaigns to specific groups or individual voters using platforms such as Facebook and Twitter (Fowler *et al.*, 2021).[1] Not only traditional campaign spending but also recent targeted campaigns are costly, and candidates need to raise funds before the election.[2]

In this paper, we consider an election contest in which candidates raise funds and then engage in campaign competition in the election. In our model, there are two candidates and a continuum of voters. The candidates try to maximize their vote shares, which captures a situation in which parties' (candidates') representation in a legislature is split proportionally to their share of votes. The two candidates have different values for the vote, and we call the candidate who has a high (resp., low) value a *strong* (resp., *weak*) candidate. This represents a case that the benefits or utilities for a candidate of being able to implement a policy or the rent from power are different across candidates, even though the legislative powers are proportionally split.

Following the convention of "redistributive" or "targetable" politics literature (e.g., Myerson, 1993; Boyer *et al.*, 2017, among others), we assume that candidates can target campaign spending at individual voters by employing a distribution of campaign spending. Each voter observes each candidate's voter-specific campaign effort, which is an independent random draw from the distribution of costly campaign spending chosen by the candidate, and then votes for the candidate who provides him with a higher level of the persuasive campaign effort. To secure the cost of campaign spending, candidates must raise funds beforehand by exerting non-monetary effort. The cost of fund-raising is sunk and cannot be recovered, so the amount of fund candidates raise will depend on their campaign spending strategies.

Thus, our model consists of two stages of decision-making. In the fund-

[1]See also Vincent and Turcotte (2018) for more details on political messaging and marketing in the 2016 US presidential election.

[2]The total amount spent in the 2020 US presidential election amounted to \$6.6 billion, and Mr. Biden's campaign committee had raised \$938 million as of October 14. See "The 2020 Campaign Is the Most Expensive Ever (By a Lot)" by Shane Goldmacher, *New York Times*, October 28, 2020, `https://www.nytimes.com/2020/10/28/us/politics/2020-race-money.html`.

raising stage, candidates simultaneously choose the level of budget by bearing costs. They then move on to the election stage at which candidates decide how to allocate their budgets via campaign spending. In this setting, we characterize an equilibrium, consisting of budget choice and budget allocation strategies. We show that the strong candidate raises more funds (i.e., chooses a higher level of budget) than the weak candidate. This is intuitive since even if candidates obtain the same share of votes, the strong candidate enjoys a higher value which makes him bear a higher level of fund-raising cost. In the subsequent election stage, the weak candidate provides zero persuasive effort with a positive probability—meaning that each voter may receive zero persuasive effort. By doing this, the candidate can save the cost of campaign spending and use it to provide a higher level of persuasive effort to compete against the strong rival.

We also discuss the role of fund-raising costs on the candidates' budget choice and allocation distributions. As noted by Hart (2016), when the cost of fund-raising is linear, our two-stage model coincides with a (complete information) all-pay auction in which each bidder pays a random bid drawn from his bid distribution. In a simpler setup, we show that when the cost of fund-raising is strictly convex, candidates now become risk averse, and this makes them choose a "more risky" distribution with lower mean spending compared to the linear cost case.

Our paper lies in the literature on electoral competition. A seminal work in this literature is Myerson (1993) in which candidates make binding promises to a continuum of voters on how they will allocate a given budget when they are elected. Candidates target individual voters by employing *offer distributions*, and each voter receives an offer drawn from the distribution. Our problem of the election stage naturally extends Myerson (1993) in the sense that we allow asymmetric values and budgets across candidates.

Our work is closely related to Boyer *et al.* (2017) and Crutzen and Sahuguet (2009), both are based on the model of Myerson (1993). In Boyer *et al.* (2017), candidates expend persuasive effort, not merely "promise", so that the campaign expenditures are paid to voters. This makes candidates endogenously choose their expenditure level. While we adopt the same framework in that candidates' campaign spending is costly, they must raise funds to cover such spending and the fund-raising itself is costly, unlike to Boyer *et al.* (2017) in which candidates can choose budget level without any cost.

In Crutzen and Sahuguet (2009), two candidates compete in terms of targeted redistributive promises as in Myerson (1993), but this must be financed through distortionary tax. Thus, candidates must decide their tax schemes and then choose offer distributions subject to the budget constraint determined by the tax scheme. Although we consider campaign spending instead of redistributive policy, candidates in the election stage in our model face a similar problem to choosing persuasive effort distribution subject to a budget constraint.

The remainder of this paper is organized as follows. In Section 2, we present our model formally. In Section 3, we provide equilibrium characterization. We characterize equilibrium via backward induction beginning with the election stage (Section 3.1) and then the fund-raising stage (Section 3.2). In Section 4, we discuss the role of cost functions and the relation between our model and all-pay auctions. Section 5 concludes the paper. Proofs are provided in the appendix unless stated otherwise.

2. MODEL

There are two candidates, 1 and 2, and a continuum of voters, which approximates a large finite number of voters, of unit measure. Candidates want to maximize their shares of votes, while they have different values for the votes. Each candidate i's value for a vote is v_i for $i = 1, 2$, and we assume that $v_1 \geq v_2$ without loss of generality. Following Boyer *et al.* (2017), candidates compete in an election via persuasive effort, and voters are ex-ante homogeneous and sensitive only to the persuasive effort. Thus, each voter votes for a candidate who expends more persuasive effort on him.

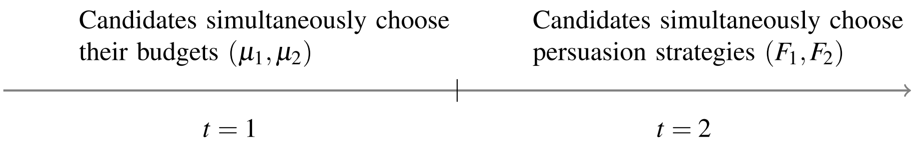
Formally, each candidate i's persuasion strategy is described by a distribution function F_i of non-negative random variable X_i for each i. If a candidate i has chosen F_i, then each voter receives an amount of persuasive effort that is an independent draw from F_i. The voter votes for candidate i if he receives a higher draw from F_i than from F_j, which is the rival candidate j's persuasion strategy for the random variable X_j. In the event of a tie, the voter is assumed to choose any candidate randomly. Thus, candidate i's expected vote share is given by

$$\text{Prob}(X_i > X_j) + \frac{1}{2}\text{Prob}(X_i = X_j). \tag{1}$$

Letting x_i and x_j denote realized values of X_i and X_j, respectively, candidate i's payoff obtained from any arbitrary voter is

$$w_i(x_i, x_j) = \begin{cases} v_i & \text{if } x_i > x_j, \\ \frac{v_i}{2} & \text{if } x_i = x_j, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Figure 1: Timing of election contest



A key feature of our model is that expending persuasive effort is costly and so candidates must raise funds before the election competition. Specifically, the cost of expending x amount of persuasive effort to a voter is x, and so the total spending of candidate i with persuasion strategy F_i is $\mathbb{E}_{F_i}[X_i]$. To secure this spending, candidates must raise funds beforehand. The fund-raising itself is costly, however. It requires a non-monetary effort of fund-raising, and the cost of raising μ amount of funds is given by $\psi(\mu)$, where ψ is a twice continuously differentiable and convex function.

Assumption 1. *$\psi(0) = \psi'(0) = 0$, and $\psi'(\mu) > 0$ and $\psi''(\mu) \geq 0$ for any $\mu > 0$.*

Assumption 1 means that the marginal cost of fund-raising is weakly increasing while the marginal cost of raising zero amount of funds is zero. This can be seen as an increasing marginal cost of administration, such as accounting or book-keeping, for managing funds.

Figure 1 illustrates the timing of our election game. In the fund-raising stage at $t = 1$, both candidates simultaneously choose the amount of funds (μ_1, μ_2), each with bearing cost $\psi(\cdot)$; and then in the election stage at $t = 2$, they simultaneously choose their persuasion strategies (F_1, F_2) under the following budget constraints:

$$\mathbb{E}_{F_i}[X_i] = \mu_i. \tag{3}$$

Thus, candidate i's ex-ante expected payoff is

$$\begin{aligned}\Pi_i &:= v_i \left[\text{Prob}(X_i > X_j) + \frac{1}{2}\text{Prob}(X_i = X_j)\right] - \psi(\mu_i) \\ &= \int_0^\infty \underbrace{\left[\int_0^\infty w_i(x_i, x_j) dF_j(x_j)\right]}_{:= u_i(x_i)} dF_i(x_i) - \psi(\mu_i). \end{aligned} \tag{4}$$

For later use, let us denote the ex-ante expected benefit obtained from vote share in the election by

$$U_i(F_i, F_j) \equiv \int_0^\infty u_i(x_i) dF_i(x_i) = \int_0^\infty \left[\int_0^\infty w_i(x_i, x_j) dF_j(x_j)\right] dF_i(x_i),$$

where $u_i(x)$ denotes the expected benefit of expending persuasive effort x to an arbitrary voter.

3. EQUILIBRIUM CHARACTERIZATION

This section provides a characterization of equilibrium. As usual, our analysis proceeds by establishing equilibrium behavior via backward induction beginning with the election stage.

3.1. ELECTION STAGE

Fix any (μ_1, μ_2) chosen by the two candidates at $t = 1$. Each candidate i's problem at $t = 2$ is choosing F_i to maximize his expected payoff subject to the budget constraint (3) for a given rival's strategy F_j. That is,

$$\max_{F_i \in \Delta(\mathbb{R}_+)} U_i(F_i, F_j) = \int_0^\infty u_i(x) dF_i(x) \text{ subject to } \int_0^\infty x dF_i(x) = \mu_i. \quad (5)$$

This problem is essentially the same as the election model of Myerson (1993), and more generally, the "Continuous General Lotto game" introduced by Hart (2008) in which two players with asymmetric budget constraints choose nonnegative random variable X and Y, respectively, to maximize

$$H(X,Y) := \text{Prob}(X > Y) - \text{Prob}(X < Y).^3$$

Our problem in the election stage extends them by allowing asymmetries for both values and budgets.

Proposition 1. *Suppose that $\mu_i \geq \mu_j$. The following strategies constitute an equilibrium of the election game.*

$$F_i^*(x) = \begin{cases} \frac{1}{2\mu_i} x & \text{if } 0 \leq x \leq 2\mu_i, \\ 1 & \text{if } x > 2\mu_i, \end{cases}$$

and

$$F_j^*(x) = \begin{cases} 1 - \frac{\mu_j}{\mu_i} + \frac{\mu_j}{2\mu_i^2} x & \text{if } 0 \leq x \leq 2\mu_i, \\ 1 & \text{if } x > 2\mu_i. \end{cases}$$

[3] Note that $\text{Prob}(X > Y) + \frac{1}{2}\text{Prob}(X = Y) = \frac{1}{2}[\text{Prob}(X > Y) + \text{Prob}(X \geq Y)] = \frac{1}{2}[\text{Prob}(X > Y) + 1 - \text{Prob}(X < Y)] = \frac{1}{2}[H(X,Y) + 1]$. Thus, maximizing vote share (1) is the same as maximizing $H(X,Y)$.

Proof. See the Appendix. □

Proposition 1 provides optimal distributions of campaign spending for given budget levels. It reveals that candidate i, who has more budget, expends persuasive effort according to a uniform distribution. That is, each voter has the same chance of receiving any level of persuasive effort. Candidate j, however, plays differently: he puts some mass at zero and places remaining densities uniformly over the same interval as candidate i. Thus, each voter has a positive probability of receiving zero persuasive effort but except this, voters have the same chance of receiving any positive level of persuasive effort. To understand this, recall that candidate j has a smaller budget than i. Thus, by providing no persuasive effort with a positive probability, he can save campaign spending and can use the saved money to provide persuasive efforts that are comparable to those provided by the rival with a higher budget. Note that when both candidates have chosen the same budget $\mu_1 = \mu_2 = 1$, then the two candidates play a symmetric distribution $F_1^*(x) = F_2^*(x) = \frac{x}{2}$, which is the same distribution obtained by Myerson (1993).

It is known in the contest literature that this problem can be solved using the duality argument. (See Crutzen and Sahuguet, 2009; Roberson and Kvasov, 2012, for instance.) To explain, write the Lagrangian associated with the problem (5) as follows:

$$\mathscr{L} = \int_0^\infty u_i(x)dF_i(x) - \lambda_i \left(\int_0^\infty x dF_i(x) - \mu_i \right) = \int_0^\infty \left[u_i(x) - \lambda_i x \right] dF_i(x) + \lambda_i \mu_i,$$

where $\lambda_i \in \mathbb{R}$ is a dual variable.[4] Note that the support of F_i must be such that any persuasive effort x in this support maximizes $\mathscr{L}$. This defines a linear relation between the persuasive efforts expended in equilibrium and the probability of winning a vote associated with this expense. That is, $u_i(x) - \lambda_i x$ is maximal and constant on the support F_i. The logic behind the linear-representation condition is intuitive and follows Crutzen and Sahuguet (2009): $u_i(x)$ is the expected benefit of expending the persuasive effort of x (and so spending x dollars), and at an optimum, this benefit must be the same as the shadow cost of the budget constraint, which is the opportunity cost of a dollar. Thus, $u_i(x) - \lambda_i x$ must be a constant in the support of F_i and be smaller than that constant outside of the support.[5]

[4] See Boyd and Vandenberghe (2004) for a formal argument.

[5] See Hwang *et al.* (2022) for a similar problem with general inequality constraints. They establish a *generalized equality of payoffs* principle stating that any equilibrium strategy of a player makes the rival indifferent in terms of the "constrained-adjusted payoff."

Now, following the literature, we consider distributions such that (i) both candidates adopt the same convex support; (ii) at most one candidate puts mass at zero (i.e., $F_i(0)F_j(0) = 0$); and (iii) they do not place mass points except zero. Then, we can write candidate i's expected payoff as

$$\int_0^\infty u_i(x)dF_i(x) = \frac{v_i}{2}F_i(0)F_j(0) + v_i \int_{(0,\bar{x}]} F_j(x)dF_i(x) = v_i \int_{(0,\bar{x}]} F_j(x)dF_i(x),$$

so the above Lagrangian function becomes

$$\mathscr{L} = \int_0^{\bar{x}} \left[v_i F_j(x) - \lambda_i x\right] dF_i(x) + \lambda_i \mu_i.$$

The aforementioned linear representation implies that $v_i F_j(x) - \lambda_i x = c_i$ for some constant c_i over the support of F_i. We then find c_i, λ_i and the support using the conditions that $F_i(0)F_j(0) = 0$, $F_i(\bar{x}) = 1$, where $\bar{x}$ is upper bound of the support, and the constraint (3). In the proof of Proposition 1, we show that the proposed distributions constitute an equilibrium.

3.2. FUND-RAISING STAGE

Given the optimal persuasion distributions from the election stage, we now induct optimal fund-raising behavior and then derive the equilibrium of the entire game. To this end, observe first that from Proposition 1, we write

$$U_i(F_i^*, F_j^*) = v_i \int_0^\infty F_j^*(x)dF_i^*(x) = \begin{cases} v_i\left(1 - \frac{\mu_j}{2\mu_i}\right) & \text{if } \mu_i \geq \mu_j, \\ \frac{v_i \mu_j}{2\mu_i} & \text{if } \mu_i < \mu_j. \end{cases}$$

Substituting this into the ex-ante expected payoff given in (4) yields

$$\Pi_1 = v_1 \left(1 - \frac{\mu_2}{2\mu_1}\right) - \psi(\mu_1) \text{ and } \Pi_2 = \frac{v_2 \mu_2}{2\mu_1} - \psi(\mu_2), \tag{6}$$

provided that $\mu_1 \geq \mu_2$.[6] In what follows, we find equilibrium (μ_1^*, μ_2^*) and will verify that it indeed holds $\mu_1^* \geq \mu_2^*$. We consider two cases separately: (i) ψ is linear, and (ii) ψ is strictly convex, since there are explicit formulae of (μ_1^*, μ_2^*) in the former case.

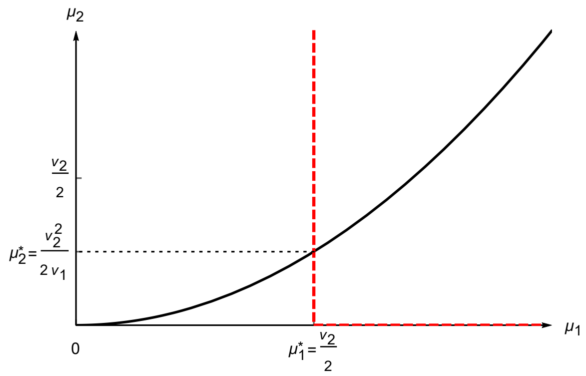
First, consider the case that ψ is a linear function. Suppose, for simplicity, that $\psi(\mu) = \mu$. Then, the candidates' expected payoffs can be written as

$$\Pi_1 = v_1 \left(1 - \frac{\mu_2}{2\mu_1}\right) - \mu_1 \text{ and } \Pi_2 = \frac{v_2 \mu_2}{2\mu_1} - \mu_2 = \left(\frac{v_2}{2\mu_1} - 1\right)\mu_2,$$

provided that $\mu_1 \geq \mu_2$.

[6] The payoffs for the case $\mu_1 < \mu_2$ can be written similarly.

Figure 2: Best responses when $\psi(\mu) = \mu$. Solid line: $BR_1(\mu_2)$, Dashed line: $BR_2(\mu_1)$.



Proposition 2. *Suppose that $\psi(\mu) = \mu$. Then, the optimal μ_1 and μ_2 are given as follows:*

$$\mu_1^* = \frac{v_2}{2} \text{ and } \mu_2^* = \frac{v_2^2}{2v_1}.$$

To understand Proposition 2, observe that Π_1 is strictly concave in μ_1. Hence, the first-order condition is necessary and sufficient, from which we obtain

$$\frac{v_1\mu_2}{2\mu_1^2} = 1 \Rightarrow \mu_1 = \sqrt{\frac{v_1\mu_2}{2}}$$

for any given μ_2. This describes candidate 1's best response against candidate 2's choice of μ_2. Next, observe also that Π_2 is linear in μ_2. Hence, for a given μ_1, candidate 2's best-response is

$$\mu_2 = \begin{cases} \infty & \text{if } v_2 > 2\mu_1, \\ [0,\infty) & \text{if } v_2 = 2\mu_1, \\ 0 & \text{if } v_2 < 2\mu_1. \end{cases}$$

Figure 2 illustrates both candidates' best-responses and finds optimal (μ_1^*, μ_2^*) as given in Proposition 2. Note that $\mu_1^* \geq \mu_2^*$ from that $v_1 \geq v_2$.[7] This is intuitive.

[7]It is easy to see that there is no equilibrium with $\mu_2 > \mu_1$. If not, then by the same argument above, we must have $\mu_2 = \frac{v_1}{2} > \mu_1 = \frac{v_1^2}{2v_2}$, which contradicts that $v_1 \geq v_2$.

Since the strong candidate (i.e., candidate 1) has a higher value for each vote than the weak candidate (i.e., candidate 2), the marginal benefit from one unit of campaign spending of the former is higher than that of the latter. Thus, the strong candidate raises more funds by bearing a higher cost than the weak rival.

Substituting the optimal (μ_1^*, μ_2^*) into the distributions obtained in Proposition 1, we have the following equilibrium for the linear cost function.

Corollary 1. *Suppose that $\psi(\mu) = \mu$. Then, there is an equilibrium in which candidate chooses $(\mu_1^*, \mu_2^*) = (\frac{v_2}{2}, \frac{v_2^2}{2v_1})$ and the following persuasion distributions:*

$$F_1^*(x) = \begin{cases} \frac{x}{v_2} & \text{if } 0 \leq x \leq v_2, \\ 1 & \text{if } x > v_2, \end{cases}$$

and

$$F_2^*(x) = \begin{cases} 1 - \frac{v_2}{v_1} + \frac{x}{v_1} & \text{if } 0 \leq x \leq v_2, \\ 1 & \text{if } x > v_2. \end{cases}$$

Next, consider the case that ψ is a strictly convex function. Observe that in this case, each Π_i, given in (6), is strictly concave in its own strategy μ_i for both $i = 1, 2$. The corresponding first-order conditions are given by

$$\frac{v_1 \mu_2}{2\mu_1^2} = \psi'(\mu_1) \text{ and } \frac{v_2}{2\mu_1} = \psi'(\mu_2). \tag{7}$$

The following proposition finds a sufficient condition for the existence of (μ_1^*, μ_2^*) such that $\mu_1^* \leq v_1$ and $\mu_2^* \leq v_2$ satisfying the above first-order conditions.

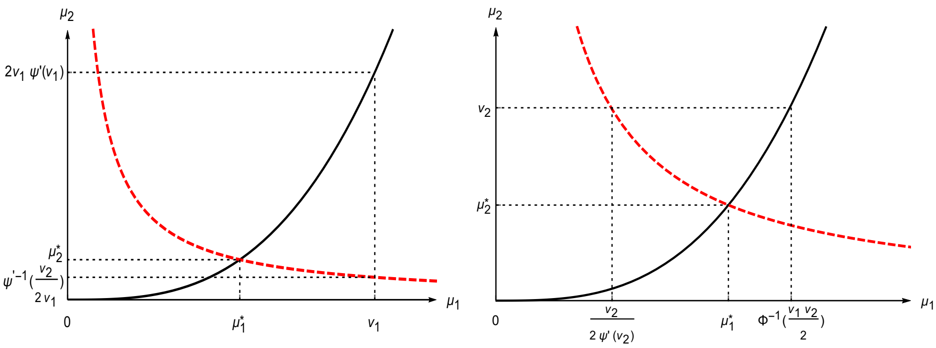
Proposition 3. *Suppose that $\psi''(\cdot) > 0$. Further, suppose that $\psi'(v_2) > \frac{1}{2}$. Then, there exists an equilibrium $(\mu_1^*, \mu_2^*) \in [0, v_1] \times [0, v_2]$ such that $\mu_1^* \geq \mu_2^*$.*

Proof. See the Appendix. □

Note that the condition $\psi'(v_2) > \frac{1}{2}$ imposed in Proposition 3 is necessary to bound the values of μ_1^* and μ_2^*. For instance, if $\psi(\mu) = 0$ for all $\mu > 0$, then it is obvious from the payoff functions in (6) that each candidate i will choose μ_i as large as possible. A sufficient condition for ψ is thus necessary to avoid such a situation. Figure 3 visualizes the candidates' best responses given by (7). Note that $\mu_1^* < v_1$ if and only if

$$2v_1 \psi'(v_1) > \psi'^{-1}\left(\frac{v_2}{2v_1}\right)$$

Figure 3: Best-Responses when $\psi''(\cdot) > 0$. Solid line: $BR_1(\mu_2)$, Dashed line: $BR_2(\mu_1)$



as in the left panel of Figure 3, and $\mu_2^* < v_2$ if and only if

$$\Phi^{-1}\left(\frac{v_1 v_2}{2}\right) > \frac{v_2}{2\psi'(v_2)},$$

where $\Phi(\mu) = \mu^2\psi'(\mu)$, as in the right panel of Figure 3. In the proof of Proposition 3, we establish the existence and uniqueness of (μ_1^*, μ_2^*) satisfying the above inequalities and further show that $\mu_1^* \geq \mu_2^*$. Below, we verify Proposition 3 using two prominent cost functions.

Example 1. *Suppose that $\psi(\mu) = \frac{1}{2}\mu^2$. Then,*

$$\mu_1^* = \frac{1}{\sqrt{2}} v_1^{\frac{1}{4}} v_2^{\frac{1}{4}} \text{ and } \mu_2^* = \frac{1}{\sqrt{2}} v_1^{-\frac{1}{4}} v_2^{\frac{3}{4}};$$

$$F_1^*(x) = \begin{cases} \frac{v_1^{-\frac{1}{4}} v_2^{-\frac{1}{4}}}{\sqrt{2}} x & \text{if } 0 \leq x \leq 2\mu_1^*, \\ 1 & \text{if } x > 2\mu_1^*, \end{cases}$$

and

$$F_2^*(x) = \begin{cases} 1 - \sqrt{\frac{v_2}{v_1}} + \frac{v_1^{-\frac{3}{4}} v_2^{\frac{1}{4}}}{\sqrt{2}} x & \text{if } 0 \leq x \leq 2\mu_1^*, \\ 1 & \text{if } x > 2\mu_1^*. \end{cases}$$

Example 2. *Suppose that $\psi(\mu) = \mu^\rho$ for $\rho > 1$. Then,*

$$\mu_1^* = \frac{1}{(2\rho)^{\frac{1}{\rho}}} v_1^{\frac{\rho-1}{\rho^2}} v_2^{\frac{1}{\rho^2}} \text{ and } \mu_2^* = \frac{1}{(2\rho)^{\frac{1}{\rho}}} v_1^{-\frac{1}{\rho^2}} v_2^{\frac{1+\rho}{\rho^2}}; \tag{8}$$

$$F_1^*(x) = \begin{cases} 2^{\frac{1-\rho}{\rho}} \rho^{\frac{1}{\rho}} v_1^{\frac{1-\rho}{\rho^2}} v_2^{-\frac{1}{\rho^2}} x & if\, 0 \leq x \leq 2\mu_1^*, \\ 1 & if\, x > 2\mu_1^*, \end{cases}$$

and

$$F_2^*(x) = \begin{cases} 1 - \left(\frac{v_2}{v_1}\right)^{\frac{1}{\rho}} + 2^{\frac{1-\rho}{\rho}} \rho^{\frac{1}{\rho}} v_1^{\frac{1-2\rho}{\rho}} v_2^{\frac{\rho-1}{\rho^2}} x & if\, 0 \leq x \leq 2\mu_1^*, \\ 1 & if\, x > 2\mu_1^*. \end{cases}$$

4. DISCUSSION

In this section, we consider an alternative model in which candidates do not raise fund beforehand but pays effort cost when they expend x amount of persuasive effort. That is, we consider a single-stage version of the election game.

In this alternative setup, candidate i's payoff from an arbitrary voter when he expends x_i and the other candidate expends x_j amounts of efforts, respectively, is given by

$$\pi_i^A(x_i, x_j) := w_i(x_i, x_j) - \psi(x_i),$$

where w_i is given by (2). Thus, the candidate's expected payoff is

$$\begin{aligned} \Pi_i^A &:= \int_0^\infty \int_0^\infty \pi_i^A(x_i, x_j) dF_j(x_j) dF_i(x_i) \\ &= \int_0^\infty \left[\int_0^\infty w_i(x_i, x_j) dF_j(x_j) - \psi(x_i)\right] dF_i(x_i). \end{aligned}$$

Note that in the baseline model, we can similarly write

$$\pi_i(x_i, x_j) = w_i(x, x_j) - \psi(\mu_i)$$

and the expected payoff

$$\begin{aligned} \Pi_i &= \int_0^\infty \int_0^\infty \pi_i(x_i, x_j) dF_j(x_j) dF_i(x_i) \\ &= \int_0^\infty \int_0^\infty w_i(x_i, x_j) dF_j(x_j) dF_i(x_i) - \psi(\mu_i) \end{aligned}$$

as given by (4).

Observe that if $\psi(\mu) = \mu$, then

$$\Pi_i^A = \Pi_i = \int_0^\infty \int_0^\infty w_i(x_i, x_j) dF_j(x_j) dF_i(x_i) - \mathbb{E}_{F_i}[X_i], \tag{9}$$

so that the alternative model coincides with the baseline model. Thus, the equilibrium of this alternative model is the same as that given by Corollary 1. As noted by Hart (2016), the vote-share maximizing game with each candidate's payoff given by (9) is strategically equivalent to a two-player all-pay auction with complete information.

For a general convex cost function $\psi(\cdot)$, the equilibrium distributions of our alternative model are given by

$$F_1^A(x) = \begin{cases} \frac{\psi(x)}{v_2} & \text{if } 0 \leq x \leq \psi^{-1}(v_2), \\ 1 & \text{if } x > \psi^{-1}(v_2), \end{cases}$$

and

$$F_2^A(x) = \begin{cases} 1 - \frac{v_2}{v_1} + \frac{\psi(x)}{v_1} & \text{if } 0 \leq x \leq \psi^{-1}(v_2), \\ 1 & \text{if } x > \psi^{-1}(v_2), \end{cases}$$

which are obtained by incorporating $\psi(\cdot)$ into the existing characterization of the all-pay auction provided by Hillman and Riley (1989) and Baye *et al.* (1996).[8]

To compare the equilibrium outcomes under the two models, let us focus on a simple environment with symmetric candidates, i.e., $v_1 = v_2 \equiv v$, and $\psi(\mu) = \mu^\rho$ for $\rho > 1$ that permits closed-form solutions. In this case, each candidate's spending under the alternative model is given by

$$\mu^A = \frac{\rho}{1+\rho} v^{\frac{1}{\rho}},$$

whereas the corresponding value under the baseline model is

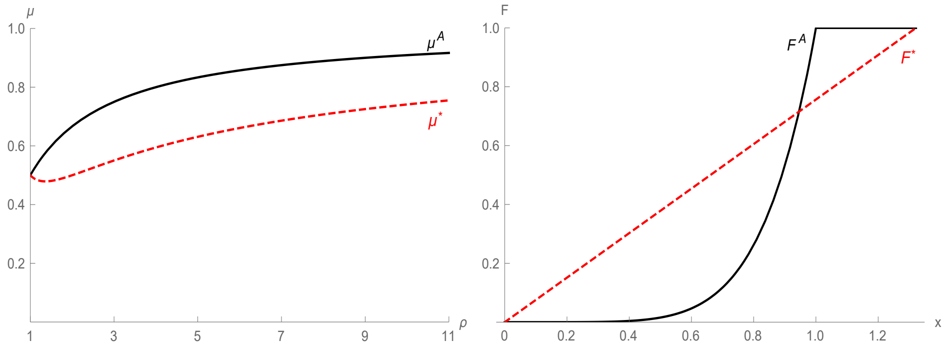
$$\mu^* = \frac{1}{(2\rho)^{\frac{1}{\rho}}} v^{\frac{1}{\rho}},$$

which is obtained from (8) in Example 2.

Figure 4 compares each candidate's spending and equilibrium persuasion distributions under the alternative and baseline models. As shown in the left panel of the figure, it holds that $\mu^A \geq \mu^*$, that is, candidates in the alternative model spend more than those in the baseline model. To understand this, note that the convex cost function makes each candidate's payoff concave, and so the candidates in the alternative model become to act as if they are risk averse. In contrast, those in the baseline model choose the expenditure level μ at first, and so at the stage of choosing the persuasion distribution F, they behave as

[8]See also Yoon (2017) for an analysis of asymmetric all-pay auctions.

Figure 4: Comparison when $v_1 = v_2 = 1$ and $\psi(\mu) = \mu^\rho$



(a) μ^A and μ^* (b) F^A and F^* and $\rho = 6$

if they are risk-neutral. This makes them choose a "more risky" distribution with a lower mean expenditure, while the candidates in the alternative model choose a "less risky" distribution with a high mean expenditure in the sense of second-order stochastic dominance. The right panel of Figure 4 shows that F^A second-order stochastically dominates F^*, i.e.,

$$\int_0^x F^*(t)dt \geq \int_0^x F^A(t)dt,$$

yielding that $\mu^A \geq \mu^*$.

5. CONCLUDING REMARKS

In this paper, we characterize the equilibrium of a two-stage election contest. We show that candidates' effort for fund-raising crucially depends on their values for the vote, but the weak candidate can compete against a strong rival by allocating persuasive effort differently from the latter. Our discussion on the cost function for fund-raising discloses the relationship between our model and all-pay auctions. There are other important aspects overlooked by the current paper, however. For example, voters may possess varying preference for candidates, and candidates may exhibit different popularity. Many countries impose restrictions on the amount of campaign spending as well as the amount of funds that candidates can raise. Although prior studies have explored similar problems in some other environments (see Che and Gale, 1998; Sahuguet and Persico, 2006, among others), examining the effects of voter heterogeneity and regularity measures on campaign spending and fund-raising in our context would be an interesting avenue.

APPENDIX

Proof of Proposition 1. First, consider candidate i and observe that

$$U_i(F_i^*, F_j^*) = v_i \int_0^\infty F_j^*(x) dF_i^*(x) = v_i \left(1 - \frac{\mu_j}{2\mu_i}\right).$$

Now, consider any F_i that satisfies the budget constraint (3). Then, candidate i's payoff from choosing such F_i deviating from F_i^* for the given rival candidate's strategy F_j^* is

$$\begin{aligned}
U_i(F_i, F_j^*) &= \int_0^\infty \int_0^{2\mu_i} w_i(x_i, x_j) dF_j^*(x_j) dF_i(x_i) = \int_0^\infty v_i F_j^*(x) dF_i(x) \\
&= \int_0^\infty v_i \min\left\{\left(1 - \frac{\mu_j}{\mu_i} + \frac{\mu_j}{2\mu_i^2}x\right), 1\right\} dF_i(x) \\
&\leq \int_0^\infty v_i \left(1 - \frac{\mu_j}{\mu_i} + \frac{\mu_j}{2\mu_i^2}x\right) dF_i(x) \\
&= v_i \left(1 - \frac{\mu_j}{\mu_i} + \frac{\mu_j}{2\mu_i^2} \int_0^\infty x dF_i(x)\right) \\
&= v_i \left(1 - \frac{\mu_j}{\mu_i} + \frac{\mu_j}{2\mu_i^2}\mu_i\right) = v_i \left(1 - \frac{\mu_j}{2\mu_i}\right) = U_i(F_i^*, F_j^*)
\end{aligned}$$

where the third equality follows from (3).

Next, for candidate j, observe that

$$U_j(F_i^*, F_j^*) = v_j \int_0^\infty F_i^*(x) dF_j^*(x) = \frac{v_j \mu_j}{2\mu_i}.$$

The candidate's payoff from choosing any F_j satisfying (3) is

$$\begin{aligned}
U_j(F_i^*, F_j) &= \int_0^\infty v_j F_i^*(x) dF_j(x) = \int_0^\infty v_j \min\left\{\frac{1}{2\mu_i}x, 1\right\} dF_j(x) \\
&\leq \int_0^\infty \frac{v_j}{2\mu_i} x dF_j(x) = \frac{v_j \mu_j}{2\mu_i} = U_j(F_i^*, F_j^*)
\end{aligned}$$

using (3) again. Thus, we have that $U_i(F_i, F_j^*) \leq U_i(F_i^*, F_j^*)$ for all $i, j = 1, 2$, showing that no candidate has an incentive to deviate. □

Proof of Proposition 3. Define

$$\mathscr{F}_1 := \frac{v_1\mu_2}{2} - \mu_1^2\psi'(\mu_1) \equiv 0 \text{ and } \mathscr{F}_2 := \frac{v_2}{2\mu_1} - \psi'(\mu_2) \equiv 0.$$

From $\mathscr{F}_1$, we have that by the Implicit Function theorem,

$$\frac{d\mu_2}{d\mu_1} = -\frac{\partial\mathscr{F}_1/\partial\mu_1}{\partial\mathscr{F}_1/\partial\mu_2} = -\frac{-2\mu_1\psi'(\mu_1) - \mu_1^2\psi''(\mu_1)}{v_1/2} > 0,$$

where the inequality holds since $\psi'(\cdot) > 0$ and $\psi''(\cdot) > 0$. Moreover, $\mu_2 = 0$ when $\mu_1 = 0$. Similarly, from $\mathscr{F}_2$, we have

$$\frac{d\mu_2}{d\mu_1} = -\frac{\partial\mathscr{F}_2/\partial\mu_1}{\partial\mathscr{F}_2/\partial\mu_2} = -\frac{-v_2/(2\mu_1^3)}{-\psi''(\mu_2)} < 0,$$

where inequality holds since $\psi''(\cdot) > 0$. Moreover, $\mu_2 \to 0$ as $\mu_1 \to \infty$ by Assumption 1. Thus, there exist a unique (μ_1^*, μ_2^*) satisfies the first-order conditions (7). See Figure 3.

We now show that $\mu_1^* \le v_1$ and $\mu_2^* \le v_2$. From Figure 3, it suffices to show that

$$2v_1\psi'(v_1) > \psi'^{-1}\left(\frac{v_2}{2v_1}\right) \iff 2v_1\psi'\big(2v_1\psi'(v_1)\big) > v_2$$

and

$$\Phi'\left(\frac{v_1v_2}{2}\right) > \frac{v_2}{2\psi'(v_2)} \iff v_1 > \frac{v_2}{2\psi'(v_2)^2}\psi'\left(\frac{v_2}{2\psi'(v_2)}\right),$$

where $\Phi(\mu) = \mu^2\psi'(\mu)$. To this end, note first that since ψ' is increasing and $\psi'(v_2) > \frac{1}{2}$, it follows that $2v_1\psi'(v_1) > v_1$. Again, since ψ' is increasing, we find that $\psi'\big(2v_1\psi'(v_1)\big) > \psi'(v_1) > \frac{1}{2}$. We thus have that

$$2v_1\psi'\big(2v_1\psi'(v_1)\big) \ge 2v_2\psi'\big(2v_1\psi'(v_1)\big) > v_2.$$

Next, since $\psi'(v_2) > \frac{1}{2}$, we have $v_2 > \frac{v_2}{2\psi'(v_2)}$, which further implies that $\psi'(v_1) > \psi'\left(\frac{v_2}{\psi'(v_2)}\right)$. Therefore, we have

$$v_1 \ge v_2 > \frac{v_2}{2\psi'(v_2)} > \frac{v_2}{2\psi'(v_2)^2}\psi'\left(\frac{v_2}{2\psi'(v_2)}\right).$$

Lastly, we show that $\mu_1^* \ge \mu_2^*$. Note that from the conditions in (7), μ_1^* and μ_2^* satisfy

$$\frac{\mu_2^*\psi'(\mu_2^*)}{\mu_1^*\psi'(\mu_1^*)} = \frac{v_2}{v_1}.$$

Since $\mu\psi'(\mu)$ is increasing in μ, we have $\mu_1^* \ge \mu_2^*$ from the fact that $v_1 \ge v_2$. $\square$

REFERENCES

Baye, M. R., Kovenock, D., and C. G. de Vries (1996). "The All-Pay Auction with Complete Information," *Economic Theory* 8, 291-305.

Boyd, S. and L. Vandenberghe (2004). *Convex Optimization,* Cambridge University Press.

Boyer, P. C., Konrad, K. A., and B. Roberson (2017). "Targeted Campaign Competition, Loyal Voters, and Supermajorities," *Journal of Mathematical Economics* 71, 49-62.

Che, Y.-K. and I. Gale (1998). "Caps on Political Lobbying," *American Economic Review* 88, 643-651.

Crutzen, B. S. Y. and N. Sahuguet (2009). "Redistributive Politics with Distortionary Taxation," *Journal of Economic Theory* 144, 264-279.

Fowler, E. F., Franz, M. M., Martin, G. J., Peskowitz, Z., and T. N. Ridout (2021). "Political Advertising Online and Offline," *American Political Science Review* 115, 130-149.

Hart, S. (2008). "Discrete Colonel Blotto and General Lotto Games," *International Journal of Game Theory* 45, 37-61.

Hart, S. (2016). "Allocation Games with Caps: from Captain Lotto to All-Pay Auctions," *International Journal of Game Theory* 45, 441-460.

Hillman, A. L. and J. G. Riley (1989). "Political Contestable Rents and Transfers," *Economics and Politics* 1, 17-39.

Hwang, S. H., Koh, Y., and J. Lu (2022). "Constrained Contests with a Continuum of Battles," Working Paper.

Myerson, R. B. (1993). "Incentives to Cultivate Favored Minorities under Alternative Electoral Systems," *American Political Science Review* 87, 856-869.

Raynauld, V. and A. Turcotte (2018). "Different Strokes for Different Folks: Implications of Voter Micro-Targeting and Appeal in the Age of Donald Trump," in *Political Marketing in the 2016 US Presidential Election*, eds., J. Gillies, Palgrave Macmillan, 11-28.

Roberson, B. and D. Kvasov (2012). "The Non-constant-sum Colonel Blotto Game," *Economic Theory* 51, 397-433.

Sahuguet, N. and N. Persico (2006). "Campaign Spending Regulation in a Model of Redistributive Politics," *Economic Theory* 28, 95-124.

Yoon, K. (2017). "On Asymmetry in All-Pay Auctions," *Journal of Economic Theory and Econometrics,* 28, 67–83.