Journal of Economic Theory and Econometrics, Vol. 36, No. 4, Dec. 2025, 71–81

Notes on the Effects of Over-Differencing on the Long-Run Relationship between Two Time Series Processes*

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Abstract We study asymptotic properties of nonparametric coherence estimator at the zero frequency in the presence of over-differenced time series. It is found that when one series is over-differenced, the coherences at the origin decay to zero, regardless of the bandwidths for kernel estimation. On the other hand, when both series are over-differenced, coherences at the zero frequency grow with the bandwidths, which lead to misleading interpretation of non-existent long-run relationships. Our simulation studies confirm the theoretical conjecture.

Keywords Coherence, long-run relationship, over-differencing, spuriousness.

JEL Classification C14, C22.

*The author is grateful to Chirok Han and seminar participants at the Allied Economic Association Annual Meeting of Korea, 2025.

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Received November 26, 2025, Revised December 12, 2025, Accepted December 12, 2025

1. INTRODUCTION

We consider a long-run relationship between two time series variables in terms of coherence, defined as the ratio of cross spectrum and the auto-spectrum, at the zero frequency. Coherence captures the degree of co-movement of the two covariance stationary series at different frequencies, where the quantity at the origin corresponds to the long-run relationship. Cointegrations are also closely related to coherences at the origin. For example, given two unit root series, the coherence of the first differenced series is equal to one (e.g., Croux, Forni and Reichlin, 2001). These inferences require correct specification of stochastic trends. In this work, we study effects of incorrect detrending on long-run relationship between the two time series variables.

It is well-known that existing unit root tests inevitably suffer from low power. Thus, there exists non-trivial probability of not rejecting the unit root null hypothesis, given that the true series follows covariance stationary process. It then causes over-differencing the data. While incorrect detrending stands as a critical issue in practical works, its possible impacts have drawn less attention in the applied or theoretical econometrics literature. Specifically, we analyze the effects of over-differencing the series on estimating long-run relationships.

In order for the coherence to be well-defined, underlying time series needs to be covariance stationary with strictly positive auto spectral density at any frequency. If series get overly differenced, both auto and cross spectrums at the origin become zero, in virtue of Cauchy-Schwarz inequality. Thus, coherence takes the form of ratio of zeros, which require higher-order expansions of spectral densities to obtain asymptotic behaviors (Velasco and Robinson, 2001). We provide these asymptotic inferences and try to verify them by a small set of simulation studies.

2. MAIN RESULTS

We begin with a simple example of over-differencing. Consider $u_t \sim iid(0, \sigma^2)$. The first differenced series $\Delta u_t = u_t - u_{t-1}$ generates the zero spectral density at zero frequency, given by

$$
\begin{aligned}
f_{\Delta u}(0) &= (2\pi^{-1})\textstyle\sum_{j=-\infty}^{\infty} R_{\Delta u}(j) = (2\pi^{-1})[R_{\Delta u}(0) + R_{\Delta u}(-1) + R_{\Delta u}(1)] \\
&= (2\pi^{-1})[2\sigma^2 - \sigma^2 - \sigma^2] = 0,
\end{aligned}
$$

where $R_{\Delta u}(j)$ is the auto-covariances of $\triangle u_t$ at the lag j. Now, consider the coherence between x and y at the zero frequency, denoted as

$$coh(0) = \frac{f_{xy}^2(0)}{f_x(0)f_y(0)},$$

where $f_{xy}(0)$, $f_x(0)$, and $f_y(0)$ denote the co-spectrum, auto-spectrum of x and y,

$$f_{xy}(0) = (2\pi)^{-1}\sum_{j=-\infty}^{\infty} R_{xy}(j),$$

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$$f_y(0) = (2\pi)^{-1}\sum_{j=-\infty}^{\infty} R_y(j),$$

and $R_{xy}(j) = E(x_t - \mu_x)(y_{t-j} - \mu_y)$, $R_x(j) = E(x_t - \mu_x)(x_{t-j} - \mu_y)$ and $R_y(j) = E(y_t - \mu_x)(y_{t-j} - \mu_y)$.

Coherence measure is valid given that both auto-spectrum $f_x(0)$ and $f_y(0)$ are strictly positive. However, if either one or two series is overdifferenced, say, $\min(f_x(0), f_y(0)) = 0$, then the cospectrum equals to zero by Cauchy-Schwarz inequality,

$$0 \leq f_{xy}^2(0) \leq f_x(0)f_y(0).$$

It follows that the $coh(0)$ takes the form of $0/0$, which makes the coherence undefined. To obtain valid asymptotic behavior in this degenerate case, it requires higher-order expansions of auto and co spectrum at the origin. The inference can be also applied to the coherency, whose squared modulus is the coherence.

Coherence is often estimated by kernel-based spectral density estimators

$$c\hat{o}h(0) = \frac{\hat{f}_{xy}^2(0)}{\hat{f}_x(0)\hat{f}_y(0)},$$

where

$$\hat{f}_{xy}(0) = (2\pi)^{-1}\sum_{j=1-T}^{T-1} k(j/M)\hat{R}_{xy}(j),$$

$$\hat{f}_x(0) = (2\pi)^{-1}\sum_{j=1-T}^{T-1} k(j/M)\hat{R}_x(j),$$

$$\hat{f}_y(0) = (2\pi)^{-1}\sum_{j=1-T}^{T-1} k(j/M)\hat{R}_y(j),$$

such that $\hat{R}_{xy}(j)$, $\hat{R}_x(j)$ and $\hat{R}_y(j)$ denote the sample versions of $R_{xy}(j)$, $R_x(j)$ and $R_y(j)$, respectively (Brockwell and Davis, 1991; Hannan, 1970). Here, $k(.)$ and M denote the kernel function and the bandwidth. Following assumptions are made.

Assumption 1. *(i)* $\int_{-\infty}^{\infty} k(x)^2 dx < \infty$;
(ii) $q \geq 2$ *for* $\{q : k_q = \lim_{x \to 0} \frac{1-k(x)}{|x|^q}\}$.

Assumption 1 corresponds to the square integrability and the degree of smoothness of $k(\cdot)$. For example, Bartlett kernel is excluded since its $q = 1$, but the QS kernel with $q = 2$ satisfies the assumption.

Assumption 2. *$M = cT^{\alpha}$, where $0 < c < \infty$, $\alpha \in (0, 1/2)$, and T is the sample size.*

Given Assumption 2, the bandwidth grows with sample size, not faster than $\sqrt{T}$-rate.

First, we consider the case that one series is over-differenced.

Proposition 1. *Suppose Assumption 1 and 2 hold. If $f_x(0) = 0$ and $f_y(0) > 0$, then the mean squared errors (MSE) of the QS kernel estimators with $q = 2$ are given by*

$$MSE(\hat{f}_{xy}(0)) = \begin{cases} O(T^{-4\alpha}), & \text{for } 0 < \alpha < 1/3, \\ O(T^{-(1+\alpha)}), & \text{for } 1/3 \leq \alpha < 1/2, \end{cases}$$

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$$MSE(\hat{f}_y(0)) = O(T^{-4\alpha}).$$

Then,

$$c\hat{o}h(0) = \begin{cases} O_p(1) & \text{for } 0 < \alpha \leq 1/5, \\ O_p(T^{(1-5\alpha)/2}) = o_p(1) & \text{for } 1/5 < \alpha < 1/3, \\ O_p(T^{(\alpha-1)/2}) = o_p(1), & \text{for } 1/3 \leq \alpha < 1/2. \end{cases}$$

The proof comes from Lee (2010, 2022), where higher-order Taylor expansions are derived for the mean squared errors of auto and cross spectral density at the zero frequency, together with the conditions on bandwidths. Detailed proofs are available upon request. Proposition 1 implies that the coherence at the origin produces a random quantity when the bandwidth grows slower than $T^{1/5}$, and it decays to zero for the bandwidth larger than $T^{1/5}$. This finding will be verified in the simulation studies given below.

In the degenerated case, there exists no trade-off between the variance and the bias of the co-spectrum and auto-spectrum at the zero frequency. Since both variance and the bias decay to zero as the sample size grows, it is not possible to obtain MSE-optimal bandwidths which are widely known in the context of HAC estimation (Andrews, 1991; Newey and West, 1994).

Next, consider the case that both series are over-differenced.

Proposition 2. *Suppose Assumption 1 and 2 hold. If* $f_x(0) = f_y(0) = 0$,

$$MSE(\hat{f}_x(0)) = O(T^{-2\alpha}), \quad MSE(\hat{f}_y(0)) = O(T^{-2\alpha}),$$

and $MSE(\hat{f}_{xy}(0))$ *is the same as that in Proposition 1. Then,*

$$c\hat{o}h(0) = \begin{cases} O_p(1), & \text{for } 0 < \alpha < 1/3, \\ O_p(T^{(3\alpha-1)}), & \text{for } 1/3 \leq \alpha < 1/2. \end{cases}$$

Unlike Proposition 1, Proposition 2 tells that the coherence at the origin grows with the bandwidth larger than $T^{1/3}$. It implies that if two series are excessivley detrended, the coherence at the zero frequency is likely to yield non-zero values, which lead to spurious interpretation for the long-run relationship between two time series variables. This finding will be verified by simulation in the next section.

3. SIMULATION STUDIES

We conduct a small set of simulations to verify the theoretical results in the previous section. Consider a covariance stationary bivariate time series:

$$x_t = \mu_1 + \alpha y_t + e_{1t} \quad \text{and} \quad y_t = \mu_2 + \beta y_{t-1} + e_{2t},$$

where errors $e_t = (e_{1t}, e_{2t})'$ is *iid* $N(0, \Omega)$ with $\Omega_{[11]} = \Omega_{[22]} = 1$, and $\Omega_{[12]} = \gamma$. Parameters are set as $\mu_1 = \mu_2 = 0$, $\alpha = 0.5$ and $\gamma = 0.2$. For the series y, the degree of persistence, $\beta = 0.9$ and 0.95 are considered, which makes the process y become very close to a unit root process. The sample size $T = 128$ and 256 are used.

From the above specification, $coh(0) = 1$ between x_t and y_t, since, from the first equation,

$$f_{xy}(0) \simeq \alpha f_y(0) \quad \text{and} \quad f_x(0) \simeq \alpha^2 f_y(0) + f_{e1}(0),$$

where $f_x(0)$, $f_y(0)$, and $f_{e1}(0)$ denote the spectral density of x, y, and e_1, at the origin, respectively. Note that $f_{e1}(0) = (2\pi)^{-1}$. Here, $f_{xy}(0)$ is the cross spectrum between x and y at the origin. Then,

$$coh(0) \simeq \frac{[\alpha f_y(0)]^2}{[\alpha^2 f_y(0) + f_{e1}(0)] \times f_y(0)} \simeq 1.$$

Given this, if we take the first difference for either x_t or both x_t and y_t, it makes the process as integrated of order -1. Then, we compute the QS Kernel estimator for coherence over the range of the bandwidth given by $M = (i/80)T^{1/5}$ for $i = 1, .., 100$.

Figures 1(a)-Figure 2(d) show the coherence estimates of over-differen-ced series, along with the true coherence, for $T = 128$ and 256. The findings are consistent with the theoretical results in the previous section. In Figures 1(a)-Figure 1(d), it is found that when only one series is over-differenced, the coherence estimates becomes very close to zero, and slowly decays to zero as the bandwidths grow. These patterns remain nearly unchanged for different sample sizes and the degrees of persistence β. Although over-differenced series are used, the coherence estimates generate near-zero values, which happens to be consistent with the fact of zero long-run relationship.

On the other hand, when both series get over-differenced, as in Figure 2(a)-Figure 2(d), the coherence estimates are shown to gradually grow with the bandwidth, which supports the theoretical results in Proposition 2. For instance, as the bandwidths reach to 3, which corresponds to $\alpha = 0.23$ and 0.2 for $T = 128$ and 256, given $M = T^{\alpha}$, the coherence estimates show the value around 0.4. These spurious values of estimates can lead to a misleading interpretation for non-existent long-run relationships between the two time series. Thus, risks of mis-specification of stochastic trends become notable when both series are over-differenced.

4. CONCLUSION

We analyze the asymptotic property of the coherence estimator based on higher-order Taylor expansion, if underlying time series is incorrectly over-differenced. In particular, when both series get over-differenced, kernel-based coherence estimators at the origin tend to grow with the bandwidth and the sample size, which entail spurious interpretations for long-run relationships between the two time series, which do not exist due to excessive detrending.

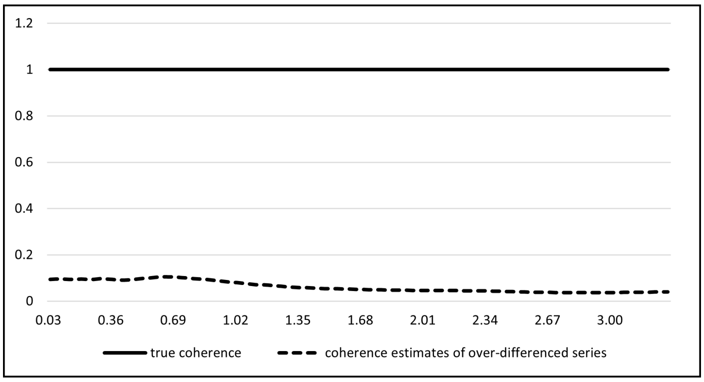

Figure 1(a): TRUE COHERENCE AND KERNEL-BASED COHERENCE ESTIMATES. The solid line denotes the true coherence and the dotted line shows the QS kernel-based coherence estimates in the case of one over-differenced series. The values of the x-axis denote the bandwidths. The sample size $T = 128$ and the degree of persistence $\beta = 0.9$.

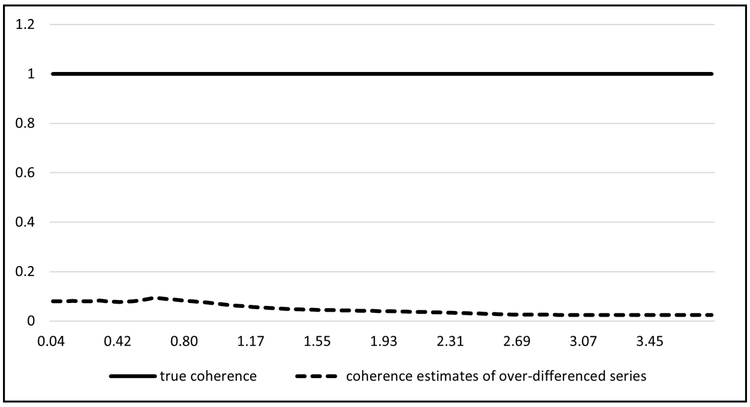

Figure 1(b): TRUE COHERENCE AND KERNEL-BASED COHERENCE ESTIMATES. The solid line denotes the true coherence and the dotted line shows the QS kernel-based coherence estimates in the case of one over-differenced series. The values of the x-axis denote the bandwidths. The sample size $T = 256$ and the degree of persistence $\beta = 0.9$.

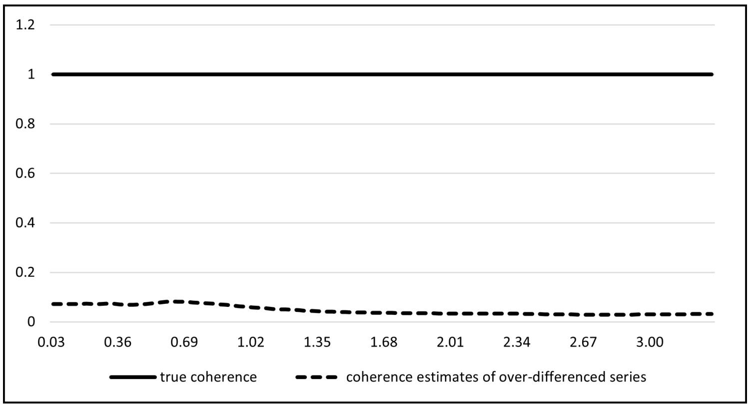

Figure 1(c): TRUE COHERENCE AND KERNEL-BASED COHERENCE ESTIMATES. The solid line denotes the true coherence and the dotted line shows the QS kernel-based coherence estimates in the case of one over-differenced series. The values of the x-axis denote the bandwidths. The sample size $T = 128$ and the degree of persistence $\beta = 0.95$.

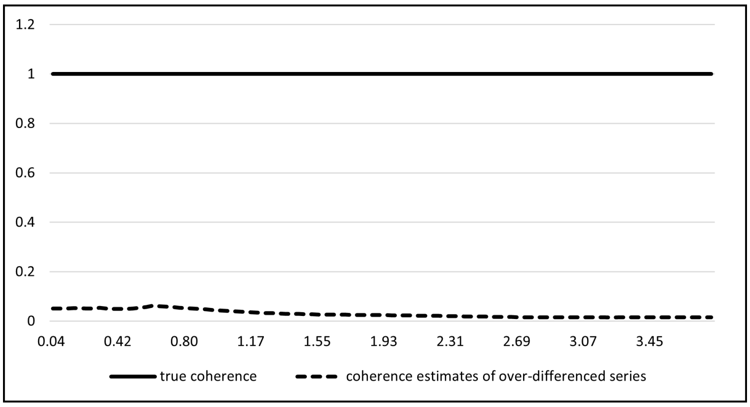

Figure 1(d): TRUE COHERENCE AND KERNEL-BASED COHERENCE ESTIMATES. The solid line denotes the true coherence and the dotted line shows the QS kernel-based coherence estimates in the case of one over-differenced series. The values of the x-axis denote the bandwidths. The sample size $T = 256$ and the degree of persistence $\beta = 0.95$.

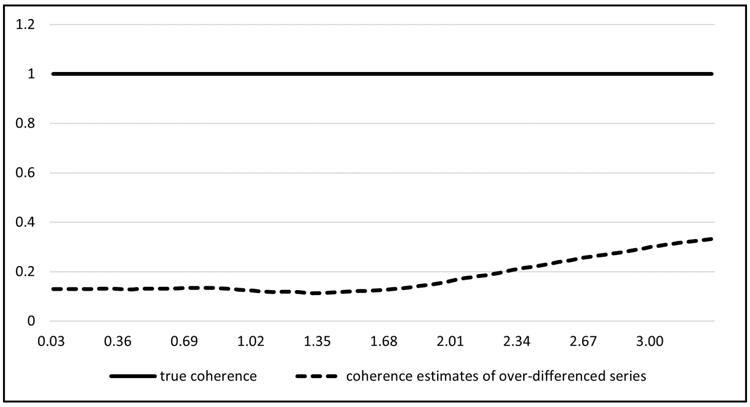

Figure 2(a): TRUE COHERENCE AND KERNEL-BASED COHERENCE ESTIMATES. The solid line denotes the true coherence and the dotted line shows the QS kernel-based coherence estimates in the case two over-differenced series. The values of the x-axis denote the bandwidths. The sample size $T = 128$ and the degree of persistence $\beta = 0.9$.

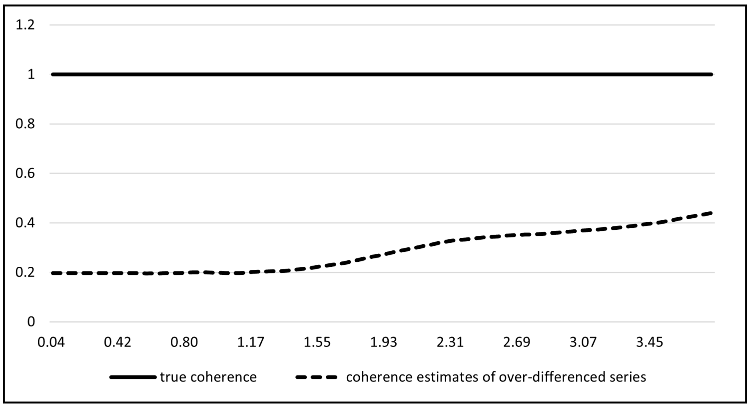

Figure 2(b): TRUE COHERENCE AND KERNEL-BASED COHERENCE ESTIMATES. The solid line denotes the true coherence and the dotted line shows the QS kernel-based coherence estimates in the case two over-differenced series. The values of the x-axis denote the bandwidths. The sample size $T = 256$ and the degree of persistence $\beta = 0.9$.

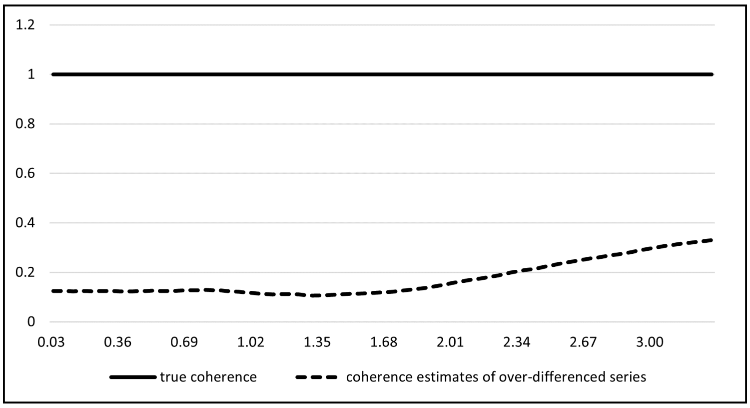

Figure 2(c): TRUE COHERENCE AND KERNEL-BASED COHERENCE ESTIMATES. The solid line denotes the true coherence and the dotted line shows the QS kernel-based coherence estimates in the case two over-differenced series. The values of the x-axis denote the bandwidths. The sample size $T = 128$ and the degree of persistence $\beta = 0.95$.

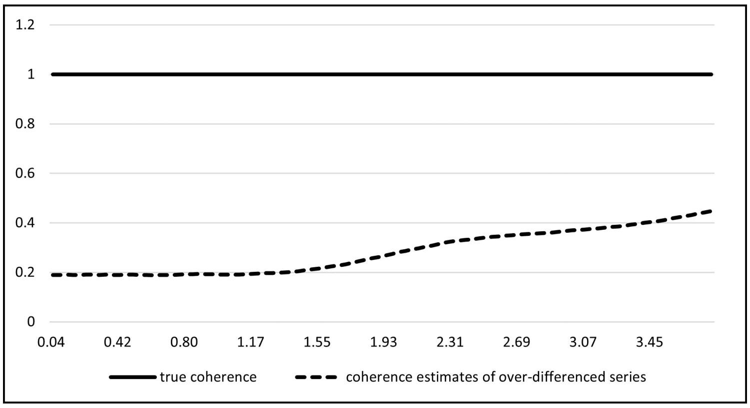

Figure 2(d): TRUE COHERENCE AND KERNEL-BASED COHERENCE ESTIMATES. The solid line denotes the true coherence and the dotted line shows the QS kernel-based coherence estimates in the case two over-differenced series. The values of the x-axis denote the bandwidths. The sample size $T = 256$ and the degree of persistence $\beta = 0.95$.

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